J. DIFFERENTIAL GEOMETRY
39 (1994) 173-213

CHERN-SIMONS PERTURBATION THEORY. II

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Abstract

In a previous paper [2], we used superspace techniques to prove that perturbation theory (around a classical solution with no zero modes) for Chern-Simons quantum field theory on a general 3-manifold M is finite. We conjectured (and proved for the case of 2-loops) that, after adding counterterms of the expected form, the terms in the perturbation theory defined topological invariants. In this paper we prove this conjecture. Our proof uses a geometric compactification of the region on which the Feynman integrand of Feynman diagrams is smooth as well as an extension of the basic propagator of the theory.

1. Introduction

In a previous paper [2], we considered the perturbative expansion for three-dimensional Chern-Simons quantum field theory about a solution A_0 to the equations of motion. We defined what we meant by the perturbative expansion and showed perturbation theory was finite. We showed that the first term in the perturbative expansion beyond the semiclassical limit defines a geometric invariant precisely in the manner one would expect based on Witten's exact solution [10]. We conjectured and gave strong evidence that the higher terms in the expansion were geometric invariants of the same type. In this paper we prove this conjecture.

More specifically, we take A_0 to be a flat connection on a principal bundle P with a compact structure group G and a closed, oriented, three-dimensional base M. We also assume that A_0 has no zero modes, i.e., that the cohomology of the exterior derivative operator $D: \Omega^*(M, \mathbf{g}) \to \Omega^{*+1}(M, \mathbf{g})$, coupled to the adjoint bundle $\mathbf{g}$ of P and A_0, vanishes. By rewriting the Lorentz gauge fixed theory as a superspace theory in [2], we were able to obtain Feynman rules that could be translated succinctly into the language of differential forms. To define the gauge fixing it was necessary to choose a Riemannian metric g on M. For $l \geq 2$, the

Received April 2, 1993 and, in revised form, April 12, 1993. This work was supported in part by the Divisions of Applied Mathematics of the U. S. Department of Energy under contracts DE-FG02-88ER25065 and DE-FG02-88ER25066.

lth-order term $I_l(M, A_0, g)$ in the perturbative expansion is a multiple integral over M^V, with $V = 2(l-1)$, of a top form depending on g. This top form, the "Feynman integrand", is smooth on the open submanifold $M_0^V \subset M^V$ consisting of the points away from all diagonals, but is singular near the diagonals. It is constructed from products of the basic "propagator" L, the integral kernel for the "Hodge theory inverse" to D. We showed that, despite the singularities, the integral defining $I_l(M, A_0, g)$ is finite. Also, we gave a "formal proof of metric independence" of $I_l(M, A_0, g)$ (ignoring the problem of products of singularities). The only dependence on the metric is therefore due to quantum field theoretic "anomalies", which arise because of the behavior of the integrand near $M^V \backslash M_0^V$.

The quantity I_l decomposes as a sum of "Feynman amplitudes" for trivalent graphs with V vertices. The nature of the anomalies is most simply stated in terms of the piece I_l^{conn} of I_l which comes from the sum over connected graphs. We conjectured, and proved for $l = 2$, that the dependence on the metric could be cancelled by subtracting a multiple of the Chern-Simons invariant for the metric connection. This conjecture is proved for all l in the present paper.

We analyzed the variation of I_2 with respect to a metric in [2] by using Stokes theorem on the differential geometric blowup of $M^2 \backslash \Delta$ along the diagonal Δ. That space $\mathrm{Bl}(M^2, \Delta)$ (see §2) has a boundary which can be identified with the tangent sphere bundle over M. To extend the argument and prove the theorem we will use a "geometric blowup" of M^V along $M^V \backslash M_0^V$. This blowup $M[V]$ is a manifold with corners and is a compactification of M_0^V to which the Feynman integrand extends smoothly. Our results can also be proved without introducing $M[V]$ by using power counting arguments of the form found in [2], but the use of $M[V]$ is more geometrical. As we will explain below, $M[V]$ is the differential geometric analog of the algebraic geometric compactification defined in [5] and [3]. Other compactifications besides $M[V]$ may also be employed to the same end, but it would take us too far afield to explain this here. In a private discussion, Kontsevich explained his use of $M[V]$ in his work on Chern-Simons perturbation theory [8]. The appearance of [5] and [3] convinced us that this approach would be the simplest.

We will also introduce an "extended propagator" $\widetilde{L}$, a vector-bundle-valued form on $(M^2 \backslash \Delta) \times \mathrm{Met}$, where Met is the space of Riemannian metrics on M. Readers worried about infinite-dimensional spaces may take Met to be any finite-dimensional submanifold of the space of metrics. Actually, for the proof of our main theorem, we could equally well proceed by taking Met to be an interval in the space of metrics. However $\widetilde{L}$ allows, among other things, an extension of the theory to families of manifolds of any dimension, as will be shown in [2]. This extension gives a mathematically precise version of the "field theory limit" of the topological open string model considered in [11]. It is also closely related to ideas of Kontsevich [8].

$\widetilde{L}$ may be expanded as a sum of its pieces $\widetilde{L}^{(d)}$ of homogeneous degree d on Met,

$$\widetilde{L} = \widetilde{L}^{(0)} + \widetilde{L}^{(1)} + \widetilde{L}^{(2)}. \tag{1.1}$$

The piece $\widetilde{L}^{(0)}$ is just the original propagator L, considered as a 2-form on $M^2 \times \text{Met}$ of degree 0 (i.e., an ordinary function) the Met directions.

As with $M[V]$, our introduction of $\widetilde{L}$ is also not strictly necessary. One could express our discussion entirely in terms of the separate components $\widetilde{L}^{(0)}$ and $\widetilde{L}^{(1)}$ of Met, without unifying them as part of a larger structure. Although introducing $\widetilde{L}$ will allow us to be more succinct, the reader may find it illuminating to make the occurrences of $L = \widetilde{L}^{(0)}$ and $\widetilde{L}^{(1)}$ explicit. This will give the arguments more in the language of [2], where $\widetilde{L}^{(1)}$ is called B.

Outline. Sections 2 and 3 are largely an exposition of parts of [2] with some extensions and modifications, along with special accommodation, we hope, to mathematicians. See [2] and references therein for more explanation of the relation to the physics literature. We review the basic propagator L and its properties in §2. In §3 we define the terms in the perturbation expansion, namely I_l and I_l^{conn}, and give the Feynman graph interpretation of these multiple integrals over M^V.

The properties of the extended propagator $\widetilde{L}$ needed in the proof of our main theorem are stated in §4.1. The actual definition of $\widetilde{L}$ and the proof of some of the properties are given in §4.2. The remaining properties, relating to the fact that $\widetilde{L}$ extends smoothly to a covariantly closed form on $\text{Bl}(M^2, \Delta) \times \text{Met}$, are proved in §4.3.

The compactification $M[V]$ is described as a closure of M_0^V in a larger topological space in §5.1. $M[V]$ is described explicitly as a point set in §5.2. A stratification of $M[V]$ is introduced in §5.3. One proof that $M[V]$ is a manifold with corners (such that the codimension k open strata of the stratification are smooth open subsets of the codimension-k boundary of $M[V]$) follows by directly mimicking the construction in [5] but using differential geometric blowups rather than algebraic geometric ones. As an alternative to this, we give an explicit atlas of coordinates on $M[V]$ in §5.4.

The results of §§4 and 5 allow us to prove the main theorem in §6.

A short appendix is included to describe our use of graded tensor product and our mathematically unusual sign conventions for push-forward integrals (which arise naturally from the superspace formulation of the field theory).

The presentations in §§4.3 and 5.4 are rather brief. Further elaboration, in the context of generalizations, will be found in a future paper by the first-named author [1].

2. Review of the basic propagator and its properties

The Feynman rules expressed in the language of differential forms use the "Hodge theory inverse" to D. This is the operator

$$\begin{aligned} D^{-1} &\equiv D^{\dagger} \circ \Delta_M^{-1} \\ &= \Delta_M^{-1} \circ D^{\dagger} : \Omega^j(M, \mathbf{g}) \to \Omega^{j-1}(M, \mathbf{g}), \qquad j = 1, 2, 3. \end{aligned} \tag{2.2}$$

Here $D^{\dagger}$ is the adjoint of D, and $\Delta_M \equiv \{D, D^{\dagger}\}$ is the associated Laplacian operator. Adjoints are defined with respect to the inner product on $\Omega^*(M, \mathbf{g})$ induced by a choice of bi-invariant inner product $\langle\ ,\ \rangle_{\mathrm{Lie}(G)}$ on the Lie algebra $\mathrm{Lie}(G)$ of G, and a choice of Riemannian metric g on M.

The operator D^{-1} can be written as an integral operator with kernel L, known as the propagator. L belongs to $\Omega^2(M_1 \times M_2, \mathbf{g}_1 \otimes \mathbf{g}_2)$ (where the subscripts 1 and 2 refer to distinct copies of M and the corresponding bundles over them), and is defined by

$$(D^{-1}\psi)_a(x) = \int_{y \in M_2} L_{ab}(x, y) \wedge \psi_b(y) \qquad \forall \psi \in \Omega^*(M, \mathbf{g}). \tag{2.3}$$

Here we have introduced the Lie algebra indices a and b which arise after introducing an orthonormal basis $\{T_a\}$ for $\mathrm{Lie}(G)$ and a local trivialization of P.[1] The totally antisymmetric structure constants f_{abc} for G are given by $[T_a, T_b] = f_{abc} T_c$.

The relation between operators and their associated integral kernels used in (2.3) is the one that arises naturally from the superspace formalism.

[1]Note that we have not used the more usual pairing $\int_{y \in M_2} L_{ab}(x, y) \wedge * \psi_b(y)$. Using the metric on $\mathrm{Lie}(G)$ to identify $\mathbf{g}_1 \otimes \mathbf{g}_2$ with $\mathrm{Hom}(\mathbf{g}_2, \mathbf{g}_1)$, $L(x, y) \wedge \psi(y)$ means to wedge the forms and apply the linear transformation from $\mathbf{g}_2$ to $\mathbf{g}_1$.

This gives an unusual sign convention in push-forward integrals like the one in (2.3). Using these sign conventions (see the appendix for more details), the relation

$$\int \psi_a \wedge (D\phi)_a = (-1)^{|\psi|+1} \int \phi_a \wedge (D\psi)_a \tag{2.4}$$

for $\psi, \phi \in \Omega^*(M, \mathbf{g})$ implies that L is antisymmetric under the involution of $\mathbf{g}_1 \otimes \mathbf{g}_2$ that exchanges $\mathbf{g}_1$ and $\mathbf{g}_2$. Equivalently, (2.4) reads

$$\int \langle \psi, D\phi \rangle_{\mathrm{Lie}(G)} = (-1)^{|\psi|+1} \int \langle \phi, D\psi \rangle_{\mathrm{Lie}(G)}. \tag{2.5}$$

General elliptic operator theory guarantees that, as a vector-bundle-valued form on M^2, L is smooth away from the diagonal $\Delta \subset M \times M$ and has singularities as one approaches Δ which are computable by an explicit local construction. Further, since all flat bundles are locally trivial, the singularity must factor as a product of the singularity for the ordinary exterior derivative times the identity operator on the Lie algebra.

In fact it turns out that L extends smoothly to a form, L_B, on the differential geometric blowup, $B_2 = BL(M^2, \Delta)$ of M^2 along Δ. B_2 is defined by replacing Δ by $S(N(\Delta))$, the sphere bundle to the normal bundle of Δ in M^2. It comes equipped with a "blowdown map" $b\colon B_2 \to M^2$. The restriction of b to the interior of B_2 is just the identity map from $M^2 \backslash \Delta$ to itself. The restriction ∂b of b to the boundary of B_2 is the bundle projection map

$$\partial b\colon \partial B_2 = S(N(\Delta)) \to \Delta. \tag{2.6}$$

This bundle is naturally isomorphic to the bundle $S(TM) \to M$.

Abusing notation, we shall denote the bundle $b^*(\mathbf{g}_i)$ for $i = 1, 2$ simply by $\mathbf{g}_i$. Then L_B belongs to the space $\Omega^2(B_2, \mathbf{g}_1 \otimes \mathbf{g}_2)$. Note that on ∂B_2, $\mathbf{g}_1 = \mathbf{g}_2$ and $\mathbf{g}_1 \otimes \mathbf{g}_2 \simeq \mathrm{Hom}(\mathbf{g}_1, \mathbf{g}_1)$.

We will show in §4 that the restriction of L_B to ∂B_2 takes the form

$$L_{B|\partial B_2} = l + (\partial b)^*(\rho), \tag{2.7}$$

where: (i) $\rho \in \Omega^2(\Delta, \mathbf{g}_1 \otimes \mathbf{g}_2)$ is smooth, and (ii) l factors as a product of a smooth ordinary form $\lambda \in \Omega^*(S(TM))$ times the identity in $\mathrm{Hom}(\mathbf{g}_1, \mathbf{g}_2) \cong \mathbf{g}_1 \otimes \mathbf{g}_2$.

The forms L_B, ρ, and λ are not only smooth, but they are also closed, as we now show. First observe that

$$\{D, D^{-1}\} = \{D, D^\dagger \circ \Delta_M^{-1}\} = \{D, D^\dagger\} \circ \Delta_M^{-1} - D^\dagger \circ [D, \Delta_M^{-1}] = 1. \tag{2.8}$$

Let D_{M^2} denote the exterior covariant derivative operator on $\Omega^*(M^2, \mathbf{g}_1, \otimes\mathbf{g}_2)$, which depends on the choice of A_0. Then the integral kernel version of (2.8) states that $D_{M^2}L$ is the kernel for the identity operator, and so is supported on the diagonal. So, the restriction of L to $M^2\backslash\Delta$ is closed as well as smooth. Since its extension L_B to B_2 is smooth, it must be closed. Hence $L_{B|\partial B_2}$ is closed. However, λ is also closed, which follows from its explicit description below (4.27). Therefore, ρ is closed as well.

The natural object that arises from the formulation of superspace perturbation theory is not the basic propagator L, but the "superpropagtor" L_s: $L_s = s(L)$ is the image of L under the linear map from $\Omega^2(M^2, \mathbf{g}_1 \otimes \mathbf{g}_2)$ to $\Omega^2(M^2, \Lambda^2(\mathbf{g}_1 \oplus \mathbf{g}_2))$ induced by the embedding

$$s: \mathbf{g}_1 \otimes \mathbf{g}_2 \to \Lambda^2(\mathbf{g}_1 \oplus \mathbf{g}_2), \tag{2.9}$$

which takes $\theta_1 \otimes \theta_2$ to $\theta_1 \wedge \theta_2$. Similarly, let

$$\rho_s = s(\rho) \in \Omega^2(\Delta, \Lambda^2(\mathbf{g}_1 \oplus \mathbf{g}_2)). \tag{2.10}$$

The antisymmetry of L under the involution exchanging $\mathbf{g}_1 \to M_1$ and $\mathbf{g}_2 \to M_2$ implies that L_s is symmetric under such an involution. That is, for $(x_1, x_2) \in M^2$, $\{j^a_{(1)}\}$ a basis of $\mathbf{g}_1$, and $\{j^a_{(2)}\}$ a basis of $\mathbf{g}_2$, we have

$$\begin{aligned} L_s(x_1, x_2) &= L_{ab}(x_1, x_2) j^a_{(1)} \wedge j^b_{(2)} \\ &= -L_{ba}(x_2, x_1) j^a_{(1)} \wedge j^b_{(2)} = L_s(x_2, x_1). \end{aligned} \tag{2.11}$$

This equation implicitly defines an identification of $\Lambda^*(\mathbf{g}_1 \oplus \mathbf{g}_2)$ with $\Lambda^*(\mathbf{g}_2 \oplus \mathbf{g}_1)$.

The Feynman integrands are built up out of the superpropagator L_s as we shall now see.

3. Formulation of perturbation theory

Fix an integer $l \geq 2$,and let $I = 3(l-1)$ and $V = 2(l-1)$. Let $M^{\{i\}}$ be the ith copy of M in the Cartesian product M^V, and $\mathbf{g}_i$ be a copy of $\mathbf{g}$ over $M^{\{i\}}$. By abuse of notation, the pullback of $\mathbf{g}_i$ by the projection map from M^V to $M^{\{i\}}$ will also be denoted $\mathbf{g}_i$. A choice of local trivialization of $\mathbf{g}_i$ determines sections $j^a_{(i)}$ of $\mathbf{g}_i$ corresponding under the trivialization to the orthonormal basis $\{T_a\}$ chosen for Lie(G). Elements of M^V will be written as $\vec{x} = (x_1, \cdots, x_V)$.

To describe the Feynman amplitude I_l for l loop perturbation theory, we introduce the bundle

$$A_V^* \equiv \Lambda^*(\mathbf{g}_1 \oplus \mathbf{g}_2 \oplus \cdots \oplus \mathbf{g}_V) \tag{3.12}$$

of Grassmann algebras over M^V. The fiber of A_V^* at a point is the graded commutative algebra generated freely by the degree one generators $\{j^a_{(i)}\,;\, i = 1, \cdots, V, a = 1, \cdots, \dim(G)\}$. The operation of interior product with the dual basis vector to $j^a_{(i)}$ will be denoted $\partial/\partial j^a_{(i)}$; this is a graded derivation of A_V^*.

For $i = 1, \cdots, V$, let $\mathrm{Tr}_i\colon A_V^* \to A_V^*$ be the map

$$\mathrm{Tr}_i \equiv \pi_i \circ f_{abc} \frac{\partial}{\partial j^a_{(i)}} \frac{\partial}{\partial j^b_{(i)}} \frac{\partial}{\partial j^c_{(i)}}, \tag{3.13}$$

where π_i is the projection operator onto the subspace of A_V^* of homogeneous degree 0 element in the $\mathbf{g}_i$ direction. The definition of Tr_i is independent of the choice of trivializations since f_{abc} is an invariant tensor. In fact it may be described more invariantly as the linear map so that

$$\mathrm{Tr}_i(\theta_1 \wedge \cdots \wedge \theta_n \wedge \omega) = \begin{cases} 0, & n \neq 3, \\ -6\langle \theta_1, [\theta_2, \theta_3] \rangle_{\mathrm{Lie}(G)}\, \omega, & n = 3, \end{cases} \tag{3.14}$$

for $\theta_1, \cdots, \theta_n$ sections of $\mathbf{g}_i$ and ω a section of A_V^* of degree 0 in the $\mathbf{g}_i$ directions.

The composition of the Tr_i acting on an element of A_V^* produces an element of overall degree 0, i.e., a real number. So acting on forms with values in A_V^*, we have a map

$$\mathrm{Tr}^{(V)} \equiv \mathrm{Tr}_1 \circ \cdots \circ \mathrm{Tr}_V\colon \Omega^*(M^V, A_V^*) \to \Omega^*(M^V). \tag{3.15}$$

The Feynman amplitude for l-loop perturbation theory may now be compactly written as

$$I_l(M, A_0, g) \equiv c_{V,I} \int_{M^V} \mathrm{Tr}^{(V)}(L^I_{\mathrm{tot}}), \qquad c_{V,I} = \frac{1}{2!^I (3!)^V V! I!}. \tag{3.16}$$

The "total propagator"

$$L_{\mathrm{tot}} \in \Omega^2(M^V, A_V^2) \subset \Omega^*(M^V, A_V^*)$$

will be defined in a moment. It makes sense to raise L_{tot} to a power since it is valued in a bundle of algebras. L^I_{tot} has degree $2I = 3V$ as a differential form, so that the integrand in (3.16) is in fact a top form on

M^V. I_l and $L_{\rm tot}$ depend on the flat connection A_0 and the metric g, since L does.

To define $L_{\rm tot}$, let

$$L_{s,\{i,j\}} \in \Omega^2(M^{(\{i,j\}}, \Lambda^2(\mathbf{g}_i \oplus \mathbf{g}_j)) \quad \text{for } i \neq j$$

be a copy of the superpropagator L_s defined on $M^{\{i,j\}}$ rather than M^2. The symmetry of L_s under involution means that the definition of $L_{s,\{i,j\}}$ is independent of whether we identify $M^{\{i,j\}}$ with $M^{\{i\}} \times M^{\{j\}}$ or $M^{\{j\}} \times M^{\{i\}}$. $L_{s,\{i,j\}}$ is smooth away from the diagonal $\Delta_{\{i,j\}} \subset M^{(\{i,j\}}$, and pulls back via the projection map $\pi_{\{i,j\}} : M^V \to M^{\{i,j\}}$ to a form

$$L_{ab}(x_i, x_j) j^a_{(i)} \wedge j^b_{(j)} = (\pi_{\{i,j\}})^*(L_{s,\{i,j\}}) \in \Omega^2(M^V, A^2_V). \tag{3.17}$$

The pullback operation here is the usual pullback of differential forms combined with the identification of the pullback of $\Lambda^*(\mathbf{g}_i \oplus \mathbf{g}_j) \to M^{\{i,j\}}$ with a subbundle of $\Lambda^*(\mathbf{g}_1 \oplus \cdots \oplus \mathbf{g}_V) = A^*_V$. Since $L_{s,\{i,j\}}$ is smooth away from the diagonal $\Delta_{\{i,j\}} \subset M^{\{i,j\}}$, the pullback is smooth away from the diagonal

$$\overline{\Delta}_{\{i,j\}} = \pi^{-1}_{\{i,j\}}(\Delta_{\{i,j\}}) \subset M^V. \tag{3.18}$$

For $i = j$, (3.17) seems not to be well defined at any point in M^V due to the singularity of L near the diagonal. It can nevertheless be given a sensible interpretation because $j^a_{(i)} \wedge j^b_{(i)}$ is antisymmetric under the exchange of a and b, whereas the singular part of L is symmetric in the Lie algebra indices. So we can interpret the singular piece as not making a contribution and define

$$L_{ab}(x_i, x_i) j^a_{(i)} \wedge j^b_{(i)} \equiv \rho_{ab}(x_i, x_i) j^a_{(i)} \wedge j^b_{(i)} \in \Omega^2(M^V, A^2_V). \tag{3.19}$$

The notation here, as in (3.17), is a useful way of summarizing a complicated pullback. That is, (3.19) can also be written as $(f_{\{i\}})^*(\rho_{s,\{i\}})$. Here $\rho_{s,\{i\}}$ is a copy of ρ_s belonging to $\Omega^2(M^{\{i\}}, \Lambda^2(\mathbf{g}_i \otimes \mathbf{g}_i))$ rather than $\Omega^2(\Delta, \Lambda^2(\mathbf{g}_1 \otimes \mathbf{g}_2))$, and $f_{\{i\}}$ is the projection map from M^V to $M^{\{i\}}$.

Finally, $L_{\rm tot}$ is given by

$$L_{\rm tot} \equiv \sum_{i,j=1}^{V} L_{ab}(x_i, x_j) j^a_{(i)} \wedge j^b_{(j)}. \tag{3.20}$$

Graphical interpretation. To obtain a graphical interpretation of (3.16), we expand

$$L^I_{\text{tot}} = \sum_{i_1, j_1=1}^{V} \cdots \sum_{i_I, j_I=1}^{V} \prod_{e=1}^{I} L_{a_e b_e}(x_{i_e}, x_{j_e}) j^{a_e}_{(i_e)} j^{b_e}_{(j_e)}. \tag{3.21}$$

A choice of i's and j's in the above sum determines a labeled,[2] oriented graph $\underline{\mathbf{G}}$ which has vertices labeled $1, \cdots, V$, edges labeled $1, \cdots, I$, and has the eth edge oriented to point from the vertex i_e to the vertex j_e $(1 \le e \le j)$. In fact, this establishes a one-to-one correspondence between the set of individual terms in the above sum and the set of labeled oriented graphs with Euler characteristic $V - I = 1 - l$. Since Tr_i vanishes on forms with degree other than 3 in the $\mathbf{g}_i$, only terms corresponding to trivalent graphs contribute to I_l. Therefore we may write

$$I_l = c_{V,I} \sum_{\substack{\underline{\mathbf{G}} \text{ trivalent} \\ \chi(\underline{\mathbf{G}}) = 1 - l}} I(\underline{\mathbf{G}}),$$

$$I(\mathbf{G}) \equiv I(\underline{\mathbf{G}}) \equiv \int_{M^V} \mathrm{Tr}^{(V)}(\mathscr{I}(\underline{\mathbf{G}})), \tag{3.22}$$

$$\mathscr{I}(\underline{\mathbf{G}}) \equiv \prod_{e=1}^{I} L_{a_e b_e}(x_{i_e}, x_{j_e}) j^{a_e}_{(i_e)} j_{(j_e)}{}^{b_e}.$$

We shall refer to $\mathscr{I}(\underline{\mathbf{G}})$ as the Feynman integrand, and $I(\mathbf{G})$ as the Feynman amplitude for $\underline{\mathbf{G}}$. In our notation for $I(\mathbf{G})$ in (3.22), we dropped the underline on $\underline{\mathbf{G}}$ since $I(\mathbf{G})$ only depends on the topological type $\mathbf{G}$ of $\underline{\mathbf{G}}$, and not on the labeling. Although this allows us to equate I_l with a sum over topological types as is usually done, it will usually be more convenient for us to stick with the formulation above.

To state our main theorem, we need the amplitude for connected graphs only:

$$I_l^{\text{conn}} \equiv c_{V,I} \sum_{\substack{\underline{\mathbf{G}} \text{ trivalent} \\ \text{connected, } l \text{ loops}}} I(\mathbf{G}). \tag{3.23}$$

Since the graphs in the sum above are connected, the Euler characteristic condition just means that the graphs have l loops.

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[2] Labelings in [2] included an ordering of the edge ends incident on any vertex. It is not necessary to include this in our labelings here, since we have not introduced explicit Lie algebra indices in our Feynman rules. Instead, our basic vertex includes a sum over orderings of incident edge ends.

4. The extended propagator $\widetilde{L}$

In this section we define the extended propagator $\widetilde{L}$ and describe its properties. The properties of $\widetilde{L}$ will be described first since that is what is used in the proof of the main theorem in §6.

4.1. Properties of $\widetilde{L}$. Let $\widetilde{TM}$, $\tilde{\mathbf{g}}_i$, $\tilde{b}$ and $\partial\tilde{b}$ be the bundles $TM \to M$ and $\mathbf{g}_i$ (over whichever base space appropriate), and the maps $b\colon B_2 \to M^2$ and $\partial b\colon \partial B_2 \to \Delta$, all trivially crossed with Met. $\nabla^{\widetilde{TM}}$ will denote the natural covariant differential on $\widetilde{TM} \to M \times \mathrm{Met}$ which is compatible with the inner product on the fibers. (At $(z, g) \in M \times \mathrm{Met}$, the inner product is simply $g(z)$.) See (4.28) for a more concrete description of $\nabla^{\widetilde{TM}}$.

The salient features of $\widetilde{L}$ are L1 through L7 below. L1–L3 simply explain what kind of object $\widetilde{L}$ is and how it is an extension of L. These properties follow immediately from the definition in §4.2. Properties L4–L7 concern the nature of the singularities of $\widetilde{L}$. They are proved in §4.3.

L1. $\widetilde{L}$ belongs to $\Omega^2(M^2 \times \mathrm{Met}, \tilde{\mathbf{g}}_1 \otimes \tilde{\mathbf{g}}_2)$.

L2. Let $\widetilde{L}^{(i)}$ be the piece of $\widetilde{L}$ of homogeneous form degree i in the Met directions. Then $\widetilde{L}^{(0)}$ equals the basic propagator L (considered as a function on Met).

L3. $\widetilde{L}$ is smooth and covariantly closed away from $\Delta \times \mathrm{Met}$.

L4. The restriction of $\widetilde{L}$ to $[M^2 \backslash \Delta] \times \mathrm{Met}$ extends smoothly to a covariantly closed form

$$\widetilde{L}_B \in \Omega^2(B_2 \times \mathrm{Met}, \tilde{\mathbf{g}}_1 \otimes \tilde{\mathbf{g}}_2). \tag{4.24}$$

L5. There are smooth closed forms

$$\tilde{\rho} \in \Omega^2(\Delta \times \mathrm{Met}, \tilde{\mathbf{g}}_1 \otimes \tilde{\mathbf{g}}_2),$$

and

$$\tilde{l} \in \Omega^2(\partial B_2 \times \mathrm{Met}, \tilde{\mathbf{g}}_1 \otimes \tilde{\mathbf{g}}_2)$$

so that

$$\widetilde{L}_B|_{\partial B_2 \times \mathrm{Met}} = \tilde{l} + (\partial\tilde{b}_2)^*(\tilde{\rho}).$$

L6. $\tilde{l}$ factors into a "manifold piece" times a "Lie algebra piece",

$$\tilde{l} = \tilde{\lambda} \otimes \mathbb{1}_{\mathbf{g}}, \quad \tilde{\lambda} \in \Omega^2(\partial B_2 \times \mathrm{Met}), \quad \mathbb{1}_{\mathbf{g}} \in \Gamma(\partial B_2 \times \mathrm{Met}, \tilde{\mathbf{g}}_1 \otimes \tilde{\mathbf{g}}_2). \tag{4.25}$$

$\mathbb{1}_{\mathbf{g}}$ is the inverse to the invariant metric on $\mathrm{Lie}(G)$ made into a bundle section. Under the identification

$$\tilde{\mathbf{g}}_1 \otimes \tilde{\mathbf{g}}_2|_{\partial B_2} = \mathrm{Hom}(\tilde{\mathbf{g}}_1, \tilde{\mathbf{g}}_2), \tag{4.26}$$

$\mathbb{1}_{\mathbf{g}}$ is the identity element on each fiber of $\tilde{\mathbf{g}}_1|_{\partial B_2}$.

L7. Identifying $\partial B_2 \to \Delta$ with $S(TM) \to M$, $\tilde{\lambda}$ may be viewed as an element of $\Omega^2(S(TM) \times \mathrm{Met})$. As such, it is given by the following local, universal formula involving the covariant differential $\nabla^{\widetilde{TM}}$ and its curvature 2-form $\tilde{\Omega} \in \Omega^2(M \times \mathrm{Met}, \mathrm{Hom}(\widetilde{TM}, \widetilde{TM}))$:

$$\begin{aligned}\tilde{\lambda}((z, \hat{u}), g) = &-\frac{1}{8\pi}\det(g_{pq}(z))^{1/2}\varepsilon_{ijk}(\hat{u}^i)\\ &\times[(d_{\mathrm{vert}}\hat{u}^j)(d_{\mathrm{vert}}\hat{u}^k) + \tilde{\Omega}^j_l(z, g)g^{lk}(z)].\end{aligned} \tag{4.27}$$

In (4.27), $\hat{u} \in S(TM)|_z$ is a vector in T_zM of unit length with respect to the inner product $g(z)$. (4.27) is written using coordinates $\{z^i\}$ about $z \in M$ and the components $\{\hat{u}^i\}$ for $\hat{u} = \hat{u}^i\partial/\partial z^i$. $d_{\mathrm{vert}}\hat{u}^i$ is the projection of $d\hat{u}^i$ onto the space of vertical 1-forms determined by ∇^{TM}.

Let $\{\partial/\partial z^i\}$ be the local trivialization of $\widetilde{TM}$ associated to the coordinates $\{z^i\}$. $\nabla^{\widetilde{TM}}$ is given by

$$\begin{aligned}&\left[\nabla^{\widetilde{TM}}_{\partial/\partial z^i}\frac{\partial}{\partial z^k}\right](z, g) = \Gamma^j_{ik}(z)\frac{\partial}{\partial z^j},\\ &\left[\nabla^{\widetilde{TM}}_m\frac{\partial}{\partial z^k}\right](z, g) = \frac{1}{2}g^{jl}(z)m_{lk}(z)\frac{\partial}{\partial z^j}\\ &\qquad\text{for } m \in T_g\mathrm{Met} = \Gamma(Sym^2(TM) \to M).\end{aligned} \tag{4.28}$$

Here $\{\Gamma^j_{ik}\}$ are the Christoffel symbols for the metric connection determined by g. The vertical projection of the function u^i of a vector (z, u) in TM is

$$d_{\mathrm{vert}}u^j = du^j + [\Gamma^j_{ik}dz^i + \frac{1}{2}(g^{-1}\delta g)^j{}_k]_z u^k.$$

$d_{\mathrm{vert}}\hat{u}^i$ in (4.27) is the value at $(z, \hat{u})$ of the pullback of $d_{\mathrm{vert}}u^j$ by the inclusion map $S(TM) \hookrightarrow TM$.

$\nabla^{\widetilde{TM}}$ can be described invariantly. Equip $M \times \mathrm{Met}$ with a Riemannian metric of the following form

$$\langle(v_1, m_1), (v_2, m_2)\rangle_{(x, g)} = g_x(v_1, v_2) + G_g(m_1, m_2), \tag{4.30}$$

for $(x, g) \in M \times \mathrm{Met}$, $v_1, v_2 \in TM_x$, $m_1, m_2 \in T\mathrm{Met}_g$. G is *any* Riemannian metric on Met (not necessarily a natural one). So $\widetilde{TM}$ is the subbundle of $T(M \times \text{Met})$ whose orthogonal complement is $M \times T\,\text{Met}$. Then $\nabla^{\widetilde{TM}}$ is the covariant differential on $M \times \text{Met}$ followed by the projection operator $\pi_{\widetilde{TM}}$ onto $\widetilde{TM}$, i.e.,

$$[\nabla^{\widetilde{TM}}_{(v,m)} w](x, g) = [\pi_{\widetilde{TM}} \circ \nabla^{T(M\times \text{Met})} w](x, g), \tag{4.31}$$

for w a section of $\widetilde{TM}$. We leave to the reader to check that this does give the connection above and to compute the curvature formulas in the next paragraph.

The curvature two-form of $\nabla^{\widetilde{TM}}$ decomposes as a sum

$$\widetilde{\Omega} = \widetilde{\Omega}^{(2,0)} + \widetilde{\Omega}^{(1,1)} + \widetilde{\Omega}^{(0,2)}, \tag{4.32}$$

where $\widetilde{\Omega}^{(i,2-i)}$ has form degree i in the M directions and $2-i$ in the Met directions. From(4.28), it follows that

$$[\widetilde{\Omega}^{(2,0)}(z, g)]^k{}_l = \left[\frac{\partial}{\partial z_i}\Gamma^k_{jl} + \Gamma^k_{im}\Gamma^m_{jl}\right]_z dz^i \wedge dz^j, \tag{4.33.1}$$

$$[\widetilde{\Omega}^{(1,1)}(z, g)]^k{}_l = \left[\delta\Gamma^k_{il} - \tfrac{1}{2}\nabla_{\partial/\partial z^i}(g^{-1}\delta g)^k{}_l\right]_z \wedge dz^i, \tag{4.33.2}$$

$$[\widetilde{\Omega}^{(0,2)}(z, g)]^k{}_l = -\tfrac{1}{4}[(g^{-1}\delta g)^k_n \wedge (g^{-1}\delta g)^n{}_l]_z. \tag{4.33.3}$$

Here $\delta\Gamma^k_{il}(z)$ and $\delta g_{ml}(z)$ are the exterior derivatives in the metric directions of the functions $\Gamma^k_{il}(z)$ and $g_{ml}(z)$ respectively. The covariant derivative operator in (4.33.2) acts on the indices k and l. This comes from the commutator of the right-hand sides of the two equation in (4.28). Note that (4.33.1) equals the usual Riemannian curvature $\Omega^k_l(z)$, considered as a function on Met. One check that the relative coefficients in (4.33.2) are correct is that the sum of the two terms is antisymmetric in k and l.

4.2. Definition of $\widetilde{L}$ and proof of L1–L3. Let W be the vector bundle $\Lambda^*(T^*(M \times \text{Met})) \otimes \mathbf{g}$ over $M \times \text{Met}$. For $g \in \text{Met}$, let $\widetilde{W}_g = \Gamma(M, W|_{M\times\{g\}})$. This may be identified with a graded tensor product $\widetilde{W}_g = \Omega^*(M, \mathbf{g})\hat{\otimes}\Lambda^*(T^*\text{Met}_g)$. $\widetilde{W}_g$ is the fiber at g of a vector bundle $\widetilde{W} \to \text{Met}$. So $\Gamma(\text{Met}, \widetilde{W}) = \Omega^*(M \times \text{Met}, \mathbf{g})$. (This may be viewed as a definition of what is meant by sections of the bundle $\widetilde{W}$ whose fibers are infinite dimensional.)

Let $D_{M\times\text{Met}}$ be the covariant exterior derivative operator on $\Omega^*(M \times \text{Met}, \mathbf{g})$, and $\widetilde{D}_M$, $\widetilde{D}_{\text{Met}}$ its pieces in the indicated directions. $\widetilde{D}_M$ can be viewed as the operator $D = D_M$ on $\Omega^*(M, \tilde{\mathbf{g}})$, made to act on the sections of $\widetilde{W}$ through its action on each fiber separately. The action $(\widetilde{D}_M)_g = D_M \hat{\otimes} \mathbb{1}_{\Lambda^*(T^*\mathrm{Met}_g)}$ on $\widetilde{W}_g$ will be abbreviated simply by D_M. Let $\kappa\colon \Omega^*(M, \mathbf{g}) \to \Omega^*(M, \mathbf{g})$ be the operator $\kappa\omega = (-1)^p\omega$ for $\omega \in \Omega^p(M, \mathbf{g})$. The operators $D^\dagger$, Hodge star $*$, and κ determine operators $\widetilde{D}^\dagger$, $\tilde{*}$, and $\tilde{\kappa}$ on $\Omega^*(M\times\mathrm{Met}, \mathbf{g})$ which are related by $\widetilde{D}^\dagger = \tilde{*}\widetilde{D}_M\tilde{*}\tilde{\kappa}$.

Define

$$\widetilde{\mathscr{O}} \equiv \{D_{M\times\mathrm{Met}}, \widetilde{D}^\dagger\}\colon \Omega^*(M\times\mathrm{Met}, \tilde{\mathbf{g}}) \to \Omega^*(M\times\mathrm{Met}, \tilde{\mathbf{g}}). \tag{4.34}$$

Then

$$\widetilde{\mathscr{O}} = \tilde{\Delta}_M + \widetilde{A}, \quad \text{where } \widetilde{A} = \{\tilde{*}\{\widetilde{D}_{\mathrm{Met}}, \tilde{*}\}, \widetilde{D}^\dagger\}. \tag{4.35}$$

Notice that $\tilde{\Delta}_M (\Delta_M$ acting on $\Omega^*(M\times\mathrm{Met}, \tilde{\mathbf{g}}))$ is a second-order elliptic operator in the M directions, $\widetilde{A}$ is a first-order operator in the M directions, and $\tilde{\Delta}_M$ and $\widetilde{A}$ both involve no derivatives in the Met directions. So $\widetilde{\mathscr{O}}$ is an operator on $\Gamma(\mathrm{Met}, \widetilde{W})$ which acts on each fiber of $\widetilde{W}$ separately. On $\widetilde{W}_g$, it acts by the elliptic operator

$$\widetilde{0}_g = \Delta_M + \widetilde{A}_g. \tag{4.36}$$

Since Δ_M is invertible and $\widetilde{A}_g$ increases form degree by 1 in the Met directions, $\widetilde{0}_g$ is also invertible. $\widetilde{\mathscr{O}}^{-1}$ is the operator on sections of $\widetilde{W}$ coming from the action $(\widetilde{\mathscr{O}}_g)^{-1}$ on the fiber $\widetilde{W}_g$ for $g \in \mathrm{Met}$.

The extended propagator $\widetilde{L} \in \Omega^*(M_1\times M_2\times\mathrm{Met}, \tilde{\mathbf{g}}_1\otimes\tilde{\mathbf{g}}_2)$ is the integral kernel for the operator

$$\widetilde{D}^\dagger \circ \widetilde{\mathscr{O}}^{-1}\colon \Omega^*(M\times\mathrm{Met}, \mathbf{g}) \to \Omega^{*-1}(M\times\mathrm{Met}, \mathbf{g}).$$

This means that

$$(\widetilde{D}^\dagger \circ O^{-1}\tilde{\psi})_a(x, g) = \int_{y\in M_2} \widetilde{L}_{a,b}(x, y, g)\wedge\tilde{\psi}_b(y, g) \quad \text{for } \tilde{\psi} \in \Omega^*(M\times\mathrm{Met}, \tilde{\mathbf{g}}), \tag{4.37}$$

or, equivalently, that

$$(D^\dagger \circ (\widetilde{\mathscr{O}}_g)^{-1}\psi)_a(x) = \int_{y\in M_2} \widetilde{L}_{ab}(x, y, g)\wedge\psi_b(y) \quad \text{for } g\in\mathrm{Met},\ \psi \in \widetilde{W}_g. \tag{4.38}$$

To describe $\widetilde{L}$ more explicitly, let $\widetilde{G} \in \Omega^3(M_1\times M_2\times\mathrm{Met}, \tilde{\mathbf{g}}_1\otimes\tilde{\mathbf{g}}_2)$ be the integral kernel for $\widetilde{\mathscr{O}}^{-1}$, defined by

$$((\widetilde{\mathscr{O}}_g)^{-1}\psi)(x) = \int_{y\in M_2} \widetilde{G}(x, y, g)\wedge\psi(y) \quad \text{for } g\in\mathrm{Met},\ \psi\in\widetilde{W}_g. \tag{4.39}$$

For fixed $g \in \mathrm{Met}$, $\widetilde{G}(\cdot, \cdot, g)$ is the integral kernel for $(\widetilde{\mathscr{O}}_g)^{-1}$. The Hadamard paramatrix construction for $\widetilde{\mathscr{O}}_g$ shows that $\widetilde{G}$ is smooth away from the diagonal and gives an explicit prescription for calculating its singularities near the diagonal. The fact that $\widetilde{G}$ is smooth in g also follows from the general construction. Thus,

$$\widetilde{L}(x, y, g) = -D_x^\dagger \widetilde{G}(x, y, g) \tag{4.40}$$

is smooth in x, y, and g away from points with $x = y$. In (4.40), $D_x^\dagger$ is the operator $D^\dagger$ acting in the directions along M_1 to which the point x belongs.

We now prove property L2 of $\widetilde{L}$. Choose $g \in \mathrm{Met}$ and $\psi \in \Omega^*(M, \mathbf{g})$, and identify $\Omega^*(M, \mathbf{g})$ with the subspace $\Omega^*(M, \mathbf{g}) \otimes \Lambda^0(T^* \mathrm{Met}_g)$ of $\widetilde{W}_g$. Let $\eta = (\widetilde{\mathscr{O}}_g)^{-1}\psi$ and η_k be the piece of η of degree k in the Met directions. Then

$$\begin{aligned} \triangle_M \eta_0 &= \psi, \\ \triangle_M \eta_k &= -\widetilde{A}_g \eta_{k-1}, \quad \text{for } k > 1. \end{aligned} \tag{4.41}$$

Hence $\eta_0 = \triangle_M^{-1}\psi$, and $D^\dagger \eta_0$ equals both $D^\dagger \circ \triangle_M^{-1}\psi$ and the piece of $D^\dagger \circ (\widetilde{\mathscr{O}}_g)^{-1}\psi$ of degree 0 in the Met directions. This means that

$$\int_{y \in M_2} \widetilde{L}^{(0)}(x, y, g) \wedge \psi(y) = \int_{y \in M_2} L(x, y; g) \wedge \psi(y)$$

for each $x \in M$, $g \in \mathrm{Met}$, and $\psi \in \Omega^*(M)$. The preceeding statement states exactly that $\widetilde{L}^{(0)}$ equals L.

Property L3 follows by generalizing (2.8). First observe that $\{D_{M \times \mathrm{Met}}, \widetilde{\mathscr{O}}\} = 0$ and so $\{D_{M \times \mathrm{Met}}, \widetilde{\mathscr{O}}^{-1}\} = 0$. Therefore

$$\{D_{M \times \mathrm{Met}}, \widetilde{D}^\dagger \circ \widetilde{\mathscr{O}}^{-1}\} = \{D_{M \times \mathrm{Met}}, \widetilde{D}^\dagger\}\widetilde{\mathscr{O}}^{-1} = \mathbb{1}_{\Omega^*(M \times \mathrm{Met}, \tilde{\mathbf{g}})}. \tag{4.42}$$

Hence $D_{M^2 \times \mathrm{Met}}\widetilde{L}$ is the integral kernel for the identity operator, and so vanishes away from $\Delta \times \mathrm{Met}$.

4.3. The extension $\widetilde{L}_B$ of $\widetilde{L}$. To prove the extension $\widetilde{L}_B \in \Omega^2(B_2 \times \mathrm{Met}, \tilde{\mathbf{g}}_1 \otimes \tilde{\mathbf{g}}_2)$ exists and satisfies properties L4–L7, we need to calculate the singularity near $\Delta \times \mathrm{Met}$ of $\widetilde{L}$. We shall use a version of the rescaling used by Getzler [6] in studying the heat kernel for generalized Laplacians to prove the local index theorem. See also [4].

Our proof will be rather condensed. Further elaboration, generalization, and discussion of the relation with heat kernels can be found in a forthcoming paper by the first author [2]. In particular, it will be shown that the restriction as a form of L_B to ∂B_2 may be derived from the equivariant Thom class obtained as a scaling of the heat kernel singularity in [9].

Throughout the discussion the metric $g \in \mathrm{Met}$ will be fixed. The space $\Lambda^*(T^*\mathrm{Met}_g)$ will be abbreviated as Λ^*_g, and we write $\mathscr{O}$ for $\widetilde{\mathscr{O}}_g$, $\overline{G}$ for the integral kernel for $\mathscr{O}$, and $\overline{L}$ for the integral kernel for $D^\dagger \circ \mathscr{O}^{-1}$. So $\overline{G}(x, y) = \widetilde{G}(x, y, g)$, $\overline{L}(x, y) = \widetilde{L}(x, y, g)$.

Since propagator singularity calculations are local and the flat connection A_0 is locally trivial, it is automatic that the singularity factorizes into a manifold piece (independent of A_0) times the identity operator on $\mathbf{g}$. Therefore we may specialize to the case where the group G has one element.

Coordinates, Taylor series, and singular series. To describe the singularity calculation we need to describe coordinates on $M_1 \times M_2$ near Δ, several gradings of the space of Λ^*_g valued forms defined near Δ, and several ways to package generalized "Taylor" series near Δ for such forms and operators acting on them.

Choose $\varepsilon > 0$ much smaller than the injectivity radius of M, and let $N = \{(z, u) \in TM;\ \|u\| < \varepsilon\}$ be the open ball of radius ε in TM. Let $E\colon N \to M_1 \times M_2$ be the map sending (z, u) to $(x, y) = E(z, u) \equiv (\exp_z u, \exp_z -u)$. E is a diffeomorphism of N onto a neighborhood of Δ in M^2. The restriction E' of E to $N' \equiv \{(z, u) \in N;\ u \neq 0\}$ is a diffeomorphism onto $E(N)\backslash\Delta$.

Given local coordinates $\{z^i\}$ on an open set U in M, define local coordinates on $N \cap TU$ by taking the coordinates of the point (z, u) to be (z^i, u^i), where (z^i) are the coordinates of z and $u = u^i\partial/\partial z^i|_z$.

Let $S = u^i\partial/\partial u^i$ be the vector field on TM generating dilation. In local coordinates $\mathscr{L}_S$ acts on $\Omega^*(N, \Lambda^*_g)$ by

$$\mathscr{L}_S = u^i \frac{\partial}{\partial u^i} + e(du^i)i\left(\frac{\partial}{\partial u^i}\right). \tag{4.43}$$

Given $\omega \in \Omega^*(N', \Lambda^*_g)$, we say that ω has total degree $|\omega|_{\mathrm{tot}}$ if $\mathscr{L}_S\omega = |\omega|_{\mathrm{tot}}\omega$. Similarly, we say that ω has degree $|\omega|_u$ in u if $u^i(\partial/\partial u^i)\omega = |\omega|_u\omega$. Finally, we say that ω has degree $|\omega|_{du}$ in du if $e(du^i)i(\partial/\partial u^i)\omega = |\omega|_{du}\omega$, i.e., if ω has form degree $|\omega|_{du}$ in the u^i directions. Equation (4.43) implies that the total degree of ω equals the degree in u plus the degree in du.

Note that the notion of u degree and du degree depend on the choice of the coordinates z^i. Properly speaking we should only talk about degree in u and du of a form on the subset of N where the coordinates $\{z^i, u^i\}$ are defined. We will not introduce any special notation for this, however, since the final results below for the propagator singularities graded by total degree are coordinate system independent. Alternatively, we could introduce covariant notions of u degree and du degree.

Suppose given smooth $\phi \in \Omega^*(M_1 \times M_2 \backslash \Delta, \Lambda_g^*)$ and smooth $\phi_s \in \Omega^s(N', \Lambda_g^*)$ for $s = s_0, s_0+1, \cdots$. We say that $\sum_{s=s_0}^{\infty} \phi_s$ is a *singular series* for ϕ if for any k, there is a K_0 so that, whenever $K \geq K_0$, $E'^*(\phi) - \sum_{s=s_0}^{K} \phi_s$ extends k times continuously differentiably across the zero section (i.e., to all of N). If $|\phi_s|_{\text{tot}} = s$ (resp. $|\phi_s|_u = s$) for all s, we say that ϕ_s is the singularity of ϕ of total degree (resp. degree in u) s. Note that the singularity of ϕ of a given degree is unique up to addition of a form smooth on all of N.

The total degree, degree in u, and degree in du of a differential operator P on $\Omega^*(N, \Lambda_g^*)$ is the amount by which it shifts the respective notions of degree, e.g.,

$$|P\omega|_{\text{tot}} = |P|_{\text{tot}} + |\omega|_{\text{tot}}.$$

Suppose Q is an order $\text{ord}(Q)$ differential operator acting on $\widetilde{W}_g = \Omega^*(M, \Lambda_g^*)$ with smooth coefficients. Let Q_x be the differential operator on $\Omega^*(M_1 \times M_2, \Lambda_g^*)$ so that $Q_x(\omega_1(x) \wedge \omega_2(y)) = (Q_x\omega_1(x)) \wedge \omega_2(y)$ for $\omega_1 \in \Omega^*(M_1, \Lambda_g^*)$, $\omega_2 \in \Omega^*(M_2)$. Q_x has a Taylor series expansion which can be described as follows. Let $E^*(Q_x)$ be the pullback of Q_x to a differential operator on $\Omega^*(N, \Lambda_g^*)$. In local coordinates

$$E^*(Q_x) = \sum_{\substack{I,J;\\ |I|+|J|\leq \text{ord}(Q)}} Q^{I,J}(z, u)\frac{\partial}{\partial z^I}\frac{\partial}{\partial u^J}, \tag{4.44}$$

where $I = (i_1, \cdots, i_k)$ and $J = (j_1, \cdots, j_l)$ are multi-indices, $|I| = k$, $|J| = l$, $\partial/\partial z^I = \frac{\partial}{\partial z^{i_1}} \cdots \frac{\partial}{\partial z^{i_k}}$, $\partial/\partial u^J = \partial/\partial u^{j_1} \cdots \partial/\partial u^{j_l}$, and $Q^{I,J}(z, u)$ is a linear transformation of $\Lambda^*(T^*(TM)_{(z,u)})\hat{\otimes}\Lambda_g^*$ depending smoothly on z and u. Let $Q^{I,J}(z, u)_{(k)}$ be the kth order term in the Taylor expansion of $Q^{I,J}(z, u)$ in the variable u. Set

$$(Q_x)_{(p)} = \sum_{\substack{I,J,k;\\ k-|J|=p}} Q^{I,J}(z, u)_{(k)}\frac{\partial}{\partial z^I}\frac{\partial}{\partial u^J}. \tag{4.45}$$

This vanishes unless $p \geq -\operatorname{ord}(Q)$. We call $\sum_p (Q_x)_{(p)}$ the Taylor series expansion of Q_x by degree in u for the following reason. If $\phi_{(p)}$ is the singularity of ϕ of degree p in u, then

(4.46) $$(Q_x\phi)_{(n)} \equiv \sum_{p,q;p+q=n} (Q_x)_{(p)}\phi_{(q)}$$

is the singularity of $Q_x\phi$ of degree n in u.

The Taylor series for Q_x may be further refined by writing

(4.47) $$(Q_x)_{(p)} = \sum_q (Q_x)_{(p,q)},$$

where $(Q_x)_{(p,q)}$ is the piece of $(Q_x)_{(p)}$ which shifts du degree by q. Also define

(4.48) $$(Q_x)_{[s]} \equiv \sum_{\substack{p,q;\\ p+q=s}} (Q_x)_{(p,q)}.$$

Then $\sum_s (Q_x)_{[s]}$ is the Taylor series expansion of Q_x by total degree; it obeys an equation similar to (4.46) but with degree in u replaced by total degree. In summary,

(4.49) $$\begin{gathered} |(Q_x)_{(p)}|_u = p, \qquad |(Q_x)_{[s]}|_{\mathrm{tot}} = s, \\ (Q_x)_{(p,q)}|_u = p, \quad |(Q_x)_{(p,q)}|_{du} = q. \end{gathered}$$

It is easy to see that the leading terms in the Taylor expansions of $\mathscr{O}_x$ and $D_x^\dagger$ by total degree are $(\mathscr{O}_x)_{[-2]}$ and $(D_x^\dagger)_{[-2]}$, respectively. In other words, $(\mathscr{O}_x)_{(p,q)}$ and $(D_z^\dagger)_{(p,q)}$ vanish for $p+q < -2$. Straightforward calculation yields

(4.50.1)
$$4(\mathscr{O}_x)_{[-2]} = -g^{ij}(z)X_iX_j + g^{jk}(z)\tilde{\Omega}(z,g)^i{}_k i\left(\frac{\partial}{\partial u^i}\right) i\left(\frac{\partial}{\partial u^j}\right),$$

(4.50.2) $$4(D_x^\dagger)_{[-2]} = -g^{ij}(z) i\left(\frac{\partial}{\partial u^i}\right) X_j,$$

where

(4.5.1) $$X_k = \frac{\partial}{\partial u^k} - \left[\Gamma^j_{ik}(z)\,dz^i + \frac{1}{2}(g^{-1}\delta g)^j{}_k\right] i\left(\frac{\partial}{\partial u^j}\right),$$

and $\tilde{\Omega}(z,g)$, $g^{-1}\delta g$ are as described in (4.33) and in what follows. The leading singularity in the expansion of $(\mathscr{O}_x)$ by degree in u is

(4.52) $$(\mathscr{O}_x)_{(-2)} = (\mathscr{O}_x)_{(-2,0)} = -\frac{1}{4}g^{ij}(z)\frac{\partial}{\partial u^i}\frac{\partial}{\partial u^j}.$$

Singularity of $\overline{G}$ and $\overline{L}$. The Hadamard parametrix construction [7] applied to the elliptic operator $\widetilde{\mathscr{O}}_g$ determines a singular series $\sum_{p=-1}^{\infty} \overline{G}_{(p)}$ for $\overline{G}$ where $|\overline{G}_{(p)}|_u = p$. The series is constructed so that $\overline{G}_{(p)}$ is of the form $\|u\|^{-1}\overline{F}_{p+1}$, where $\|u\| = g_z(u, u)^{1/2}$ and $\overline{F}_{p+1} \in \Omega^*(N, \Lambda_g^*)$ depends smoothly on z and is a polynomial of degree $p+1$ in its dependence on u. (Hadamard's construction uses the map $(z, u) \mapsto (z, \exp_z u)$ rather than E, but the results immediately translate into the packaging used here.)

The leading singularity $\overline{G}_{(-1)}$ is

$$\overline{G}_{(-1)} \equiv \frac{1}{24\pi\|u\|}\sqrt{\det(g_{lm}(z))}\varepsilon_{ijk}\, du^i\, du^j\, du^k. \tag{4.53}$$

$\overline{G}_{(p)}$ is then determined inductively in p from the fact that $\mathscr{O}_x\overline{G}(x, y) = 0$ for (x, y) away from Δ. The singular piece of this equation of degree n in u is $\sum_{k+l=n}(\mathscr{O}_x)_{(k)}\overline{G}_{(l)} = 0$. In other words

$$(\mathscr{O}_x)_{(-2)}\overline{G}_{(-1)} = 0, \tag{4.54.1}$$

$$(\mathscr{O}_x)_{(-2)}\overline{G}_{(p)} = \sum_{-1\le l\le p-1} (\mathscr{O}_x)_{(-2+p-l)}\overline{G}_{(l)} \quad \text{for } p \ge 0. \tag{4.54.2}$$

(4.54.1) follows because $G_{(-1)}$ is the flat space propagator. Equation (4.54.2) is an algebraic equation for the polynomial $\overline{F}_{p+1}$. Ellipticity of $\mathscr{O}$ implies that this equation has a unique solution.[3]

Let $\overline{G}_{(p,r)}$ be the piece of $\overline{G}_{(p)}$ of degree r in du. The piece of (4.54.2) of degree r in du is

$$(\mathscr{O}_x)_{(-2,0)}\overline{G}_{(p,r)} = \sum_{-1\le l\le p-1} \sum_q (\mathscr{O}_x)_{(-2+p-l,q)}\overline{G}_{(l,r-q)}, \quad \text{for } p \ge 0,\ 0 \le r \le 3. \tag{4.55}$$

Now we show that $\overline{G}_{(p,r)} = 0$ for $p + r < 2$ by induction on p. For $p = -1$, the result follows from (4.53). For $p \ge 0$, $p+r < 2$, it suffices to show that the right-hand side of (4.55) vanishes (since $\overline{G}_{(p,q)}$ is determined uniquely by (4.55)). This follows since $p + r < 2$ implies either $-2 + p - l + q < -2$, and so $\mathscr{O}_{(-2+p-l,q)} = 0$, or else $l + r - q < 2$, and so $\overline{G}_{l,r-q} = 0$ by the inductive hypothesis.

[3]For a general elliptic operator, the F's might also depend on powers of $\ln(\|u\|)$. No such powers appear here because, using a covariant grading rather than the coordinate dependent grading, $(\mathscr{O}_x)_{(-1)}$ vanishes.

Let $\overline{G}_{[s]} \equiv \sum_{p,q;p+q=s} \overline{G}_{(p,q)}$ be the piece of the singularity of $\overline{G}$ of total degree s. The result of the last paragraph yields that $\overline{G}_{[s]}$ vanishes for $s < 2$. Equations (4.53) and (4.55) imply that $\{\overline{G}_{[s]}, s \geq 2\}$ is uniquely determined by the following conditions:

U1. $\overline{G}_{(-1,3)}$, the piece of $\overline{G}_{[2]}$ of degree 3 in du, is given by the right-hand side of (4.53).

U2. $\|u\|\overline{G}_{[s]}(z, u)$ is a polynomial in its dependence on u.

U3. Away from $u = 0$,

$$(\mathscr{O}_x)_{[-2]}\overline{G}_{[2]} = 0, \tag{4.56.1}$$

$$(\mathscr{O}_x)_{[-2]}\overline{G}_{[s]} = \sum_{2\leq t<s} (\mathscr{O}_x)_{[-2+s-t]}\overline{G}_{[t]} \quad \text{for } s > 2. \tag{4.56.2}$$

We need only the explicit formula for $\overline{G}_{[2]}$:

$$\begin{aligned}\overline{G}_{[2]}(z, u) = \frac{1}{24\pi\|u\|}\sqrt{\deg(g_{lm}(z))}\varepsilon_{ijk}[&d_{\mathrm{vert}}u^i\, d_{\mathrm{vert}}u^j\, d_{\mathrm{vert}}u^k \\ &-3\|u\|^2\widetilde{\Omega}^i{}_k g^{kj}\, d_{\mathrm{vert}}u^k].\end{aligned} \tag{4.57}$$

Since the right-hand side obviously satisfies U1 and U2, one need only check (4.56.1) to verify (4.57). This follows by substituting (4.57) and (4.50.1) into (4.56.1) and calculating.

Since $\overline{L}(x, y) = -D_x^\dagger\overline{G}(x, y)$, $\overline{L}$ has a singular series graded by total degree of the form $\sum_{s=0}^{\infty} \overline{L}_{[s]}$, where

$$\overline{L}_{[s]} = -\sum_{-2\leq t\leq s-2} (D_z^\dagger)_{[t]}(\overline{G})_{[s-t]}. \tag{4.58}$$

Furthermore $\|u\|^3\overline{L}_{[s]}$ depends polynomially on u. This follows because $\mathrm{ord}(D^\dagger) = 1$ and $\|u\|\overline{G}_{[s]}$ is a polynomial in u.

Using (4.50.2) and (4.57) to evaluate (4.58) for $s = 0$, we find

$$\begin{aligned}\overline{L}_{[0]} &= -(D_x^\dagger)_{[-2]}(\overline{G})_{[2]} \\ &= -\frac{1}{8\pi}\det(g_{pq}(z))^{1/2}\varepsilon_{ijk}(\hat{u}^i) \\ &\quad \times [(d_{\mathrm{vert}}\hat{u}^j)(d_{\mathrm{vert}}\hat{u}^k) + \overline{\Omega}^j{}_l(z, g)g^{lk}(z)],\end{aligned} \tag{4.59}$$

where $\hat{u} = u/\|u\|$. This has exactly the same form as the right-hand side of (4.27).

Extension to B_2. Identify N' with $S(TM) \times (0, \varepsilon)$ via the map

$$N' \ni (z, u) \mapsto ((z, \hat{u}), \|u\|) \in S(TM) \times (0, \varepsilon). \tag{4.60}$$

Let $E_B\colon S(TM)\times[0,\varepsilon)\to B_2$ be the map

$$(4.61)\qquad ((z,\hat{u}),r)\mapsto\begin{cases}(z,u\hat{u})\in\partial B_2, & r=0,\\ E(z,r\hat{u})\in M^2\backslash\Delta=B^2\backslash\partial B_2, & R>0.\end{cases}$$

E_B is a diffeomorphism onto an open neighborhood of ∂B_2 (by definition of the differentiable structure on B_2). The restriction of E_B to $N'\cong S(TM)\times(0,\varepsilon)$ agrees with E'; and $E_B|_{S(TM)\times\{0\}}$ is a diffeomorphism of $S(TM)\times\{0\}$ with ∂B_2.

Observe that

$$(4.62)\qquad \overline{L}_{[s]}=\begin{cases}\|u\|^s D_s, & s=0,\\ \|u\|^s D_s+\|u\|^{s-1}d(\|u\|)E_{s-1}, & s>0,\end{cases}$$

where D_s and E_{s-1} are polynomials in $\hat{u}^i$ and $d\hat{u}^i$ (whose coefficients are smooth forms in z) of degree s and $s-1$ respectively. So $\overline{L}_{[s]}$ extends smoothly to $S(TM)\times[0,\varepsilon)$; $\overline{L}_{[s]}|_{S(TM)\times\{0\}}$ vanishes for $s>0$; $\overline{L}_{[0]}|_{S(TM)\times\{0\}}$ is given by the right-hand side of (4.27).

That $\sum_{s=0}^{\infty}\overline{L}_{[s]}$ is a singular series for $\overline{L}$ means that there are forms $\overline{\rho}_K\in\Omega^*(E(N),\Lambda_g^*)$ which become arbitrarily differentiable for large K so that

$$(4.63)\qquad (E')^*(\overline{L})=(E')^*(\overline{\rho}_K)+\sum_{s=0}^{K}L_{[s]}.$$

This implies that $\overline{\rho}\equiv(\overline{\rho}_K)|_\Delta$ is independent of K and hence smooth. $\overline{\rho}$ is the restriction (as a bundle section) of a smooth form $\tilde{\rho}\in\Omega^2(\Delta\times\mathrm{Met})$ to $\Delta\times\{g\}$. By the results of the last paragraph, $(E')^*(\overline{L})$ extends smoothly to $S(TM)\times[0,\varepsilon)$ and has restriction to $\partial B_2=S(TM)\times\{0\}$ equal to

$$(4.64)\qquad [\tilde{\lambda}+(\partial\tilde{b}_2)^*(\tilde{\rho})]|_{\partial B_2\times\{g\}}.$$

Using E_B to identify $S(TM)\times[0,\varepsilon)$ with a neighborhood of ∂B_2 in B_2 and using the smoothness of $\overline{L}$ and $\overline{L}_{[s]}$ in their dependence on Met, we see that $\tilde{L}$ extends to a smooth form $\tilde{L}_B\in\Omega^2(B_2\times\mathrm{Met})$ whose restriction to $\partial B_2\times\mathrm{Met}$ is $\tilde{\lambda}+(\partial\tilde{b}_2)^*(\tilde{\rho})$. Since we have already shown that $\tilde{L}$ is closed and direct calculation shows $\tilde{\lambda}$ is closed, it follows that $\tilde{\rho}$ and $\tilde{L}_B$ are closed.

We have now shown L4–L7 when the group G is a point. For general G the only change needed in the above discussion is that all forms become $\mathbf{g}_1\otimes\mathbf{g}_2$ valued and the singularity $\overline{L}_{[s]}$ gets multiplied by (the pullback by

E of) the $\mathbf{g}_1 \otimes \mathbf{g}_2 = \mathrm{Hom}(\mathbf{g}_2, \mathbf{g}_1)$-valued tensor whose value at $(x, y) \in E(N)$ is the parallel transport homomorphism along the short geodesic from x to y.

5. The compactification $M[V]$

In this section we will define a compactification $M[V]$ of

$$M_0^V \equiv M^V \setminus \bigcup \overline{\Delta}_{\{i,j\}}$$

and describe some of its properties. As mentioned in the introduction, $M[V]$ is a manifold with corners. That is, it is locally modeled on the space $C_n \equiv \{(t_1, \cdots, t_n) \in \mathbb{R}^n; t_i \geq 0\}$ (where $n = \dim(M[V]) = 3V$) with smooth overlap maps. Smooth maps between open sets in C_n are maps that extend smoothly to open neighborhoods in $\mathbb{R}^n$. We will denote by $\partial_k M[V]$ the "codimension-k boundary" of $M[V]$, that is, the points in $M[V]$ with at least k coordinates vanishing. So $\partial M[V] = \partial_1 M[V]$ is the full boundary. $\partial_k M[V]$ is not a manifold, but $\partial_k M[V] \setminus \partial_{k+1} M[V]$ is a disjoint union of smooth pieces, the codimension-k open strata, as we shall see. The reason $\partial_k M[V]$ is not smooth is that the closed codimension-k strata have common boundaries. (Think of the edges of a cube, which are the intersections of the face; or the vertices of a cube, which are the intersections of the edges.)

There are several equivalent definitions of $M[V]$ which can be made by taking the definitions in the algebrogeometric context of [5] and replacing algebraic geometric blowups with differential geometric blowups, i.e., replacing projective spaces by spheres. We will not give a complete treatment extending [5] to the differential geometric case. But we will describe $M[V]$ and the different strata explicitly as point sets and present coordinate charts that give $M[V]$ a structure of manifold with corners. Our goal here will be to be explicit, rather than to provide all details in proofs since an extension of the blowup procedure in [5] to manifolds with corners gives a simple conceptual proof. To perform the anomaly calculation in §6, we use Stokes theorem; for this all we really need are the coordinates on the codimension-1 open strata.

5.1. Definition of $M[V]$ as a closure. For the remainder of this section, the integer V will be fixed. In accordance with our application to Feynman graphs, elements of the set $\underline{V} \equiv \{1, \cdots, V\}$ will be referred to as vertices. The set M^V is by definition $M^{\underline{V}}$, the set of maps from $\underline{V}$ to M. For S a subset of $\underline{V}$ containing at least two vertices, Δ_S will denote the smallest diagonal in $M^S = \mathrm{Map}(S, M)$ consisting of constant maps from S to M. Similarly, $\overline{\Delta}_S \subset M^{\underline{V}}$ will denote the diagonal in $M^{\underline{V}}$ which maps to Δ_S under the projection map from $M^{\underline{V}}$ to M^S. $\overline{\Delta}_S$ consists of maps from $\underline{V}$ to M which send all vertices in S to the same point in M.

The blowup of M^S along the diagonal Δ_S will be called $\mathrm{Bl}(M^S, \Delta_S)$. It has interior $M^S \backslash \Delta_S$ and boundary $S(N(\Delta_S \subset M^S))$, the sphere bundle of the normal bundle to the small diagonal in M^S. This differential geometric blowup distinguishes a direction in $N(\Delta_S)$ from its negative. Let $\mathrm{Bl}_a(M^S, \Delta_S)$ denote the algebraic geometric blowup used in [5]. There is a natural map $\phi_S : \mathrm{Bl}(M^S, \Delta_S) \to \mathrm{Bl}_a(M^S, \Delta_S)$ which identifies two rays in $N(\Delta_S)$ in opposite directions.

Since the projection map $\pi_S : M^{\underline{V}} \to M^S$ maps M_0^V to the interior of $\mathrm{Bl}(M^S, \Delta_S)$ for $S \subset \underline{V}$ with $|S| \geq 2$, it determines a map $\pi_{0,S} : M_0^V \to \mathrm{Bl}(M^S, \Delta_S)$. Putting these maps together with the inclusion $\vec{f}_0 : M_0^V \to M^{\underline{V}}$, we obtain an embedding

$$M_0^V \subset M^{\underline{V}} \times \prod_{|S| \geq 2} \mathrm{Bl}(M^S, \Delta_S). \tag{5.65}$$

The space on the right-hand side of (5.65) will be called $\mathscr{B}$. Since $\mathscr{B}$ is a product of manifolds with boundary, it is a manifold with corners. $M[V]$ is defined to be the closure of the image of M_0^V in $\mathscr{B}$. In the algebrogeometric context, there is a corresponding space

$$\mathscr{B}_a = M^{\underline{V}} \times \prod_{|S| \geq 2} \mathrm{Bl}_a(M^S, \Delta_S) \tag{5.66}$$

and a continuous map $\phi : \mathscr{B} \to \mathscr{B}_a$. The map ϕ sends $M[V]$ onto the Fulton-Macpherson compactification $M_a[V]$, the closure of M_V^0 in $\mathscr{B}_a$.

In [5], $M_a[V]$ is shown to be equal to a sequence of algebrogeometric blowups of M^V. When M is a nonsingular, the blowups are on smooth submanifolds and hence $M_a[V]$ is a smooth manifold, in fact a submanifold of $\mathscr{B}_a$. This procedure carries over to the differential geometric setup using manifolds with corners, so that $M[V]$ is equal to a succession of blowups of M^V along submanifolds with corners and is a submanifold with corners of $\mathscr{B}$.

We now describe $M[V] \subset \mathscr{B}$ explicitly. A point in $\mathscr{B}$ is of course a pair $(\vec{x}, \{\vec{x}_{B,S}\})$, where $\vec{x}$ is an element of $M^{\underline{V}}$, and $\vec{x}_{B,S}$ is an element of $\mathrm{Bl}(M^S, \Delta)$ for each $S \subset \underline{V}$ with $|S| \geq 2$. Given such a pair, let $\vec{x}_S$ be the image of $\vec{x}_{B,S}$ under the blowdown map from $\mathrm{Bl}(M^S, \Delta_S)$ to M^S.

If $\vec{x}_S$ does not lie in Δ_S, $\vec{x}_{B,S}$ just equals $\vec{x}_S$. Otherwise $\vec{x}_{B,S}$ also contains the information of a point in the fiber of $S(N(\Delta_S \subset M^S))$ at $\vec{x}_S$.

Given $\vec{x}_S \in \Delta_S$, let $z \in M$ be the common location of all the vertices in S. The fiber $N(\Delta_S \subset M^S)|_{\vec{x}_s}$ may be identified with $[T_zM]^S/T_zM$, the quotient of the set of maps from S to T_zM by overall translations. The sphere bundle is then the further quotient of the set of nonzero elements of the normal bundle by the group $\mathbb{R}_+$ of dilations. Given a point $\vec{u}_S \in [T_zM]^S$, its orbit under the combined actions of T_zM and $\mathbb{R}_+$ will be written $[\vec{u}_S]$. So $S(N(\Delta_S \subset M^S))|_{\vec{x}_S}$ is the set of orbits $[\vec{u}_S]$ such that not all of the components of $\vec{u}_S$ are the same. In the terminology of [5], $[\vec{u}_S]$ is called a *screen* for S at z. Given a metric on M, screens may be uniquely represented by vectors $\vec{u}_S$ chosen to have norm 1 and to be orthogonal to Δ_S. It will be convenient to set $u_{S,j} = 0 \in TM_{x_j}$ when $j \notin S$, so that now $\vec{u}_S \in T_{\vec{x}}M^{\underline{V}}$ has norm 1 and is orthogonal to $\overline{\Delta}_S$.

5.2. Description of $M[V]$ as a point set. Which points in $\mathscr{B}$ lie in $M[V]$? Let $\mathscr{C}$ be the subset of $\mathscr{B}$ consisting of points $(\vec{x}, \{\vec{x}_{B,S}\})$ satisfying the following two conditions:

C1: $\vec{x}_S = \vec{x}|_S$ for $S \subset \underline{V}$, $|S| \geq 2$.

C2: *Compatibility condition for screens.* Suppose that S' is a subset of S with $|S'| > 1$, $\vec{x}$ maps all vertices in S to z, and the values of $\vec{u}_S$ on the vertices in S' are not all equal. Then $[\vec{u}_{S'}]$ equals the restriction, $[\vec{u}_S|_{S'}]$, of the screen for S to a screen for S'.

We now sketch an argument showing $M[V] = \mathscr{C}$. Since condition C1 holds for points in $M_0^V \subset \mathscr{B}$, it holds for $M[V]$. Since $\vec{x}_S$ is determined by $\vec{x}$ for points in $M[V]$, we may consider $M[V]$ to be a set of pairs $(\vec{x}, \{[\vec{u}_S]; \vec{x}|_S \in \Delta_S\})$.

Suppose $\vec{x}(t)$ is a smooth path in $M^{\underline{V}}$ parametrized by t in $\mathbb{R}_{\geq 0}$ (the non-negative reals) with the property that $\overline{x}(0) = (z, z, \cdots, z) \in M^{\underline{V}}$ and $\vec{x}(t) \in M_0^{\underline{V}}$ for $t > 0$. Choose local coördinates on M with the origin centered about z. The Taylor expansion of the components of $\overline{x}(t)$ about $t = 0$ takes the form

$$x_i(t) = v_i(1)t + v_i(2)t^2 + \cdots, \quad \text{for } i \in \underline{V}. \tag{5.67}$$

Although it is by a coordinate-system-dependent operation, the components of $v_i(k)$ determine a vector in T_zM. Let $n(S)$ be the smallest integer so that $v_i(n(S)) \neq v_j(n(S))$ for some $i, j \in S$. Suppose now that the path $\vec{x}(t)$ is chosen so that $n(S) < \infty$ for all S, $|S| > 1$. Then the limit

$$(\vec{x}_S, [\vec{u}_S]) \equiv \lim_{t\to 0^+} \pi_{0,S}(\vec{x}(t)) \tag{5.68}$$

exists; $\vec{x}_S$ maps every vertex in S to z, and $\vec{u}_S$ is the map that sends the vertex $i \in S$ to $v_i(n(S))$.

The hypothesis in the compatibility condition for screens which requires that the values of $\vec{u}_S$ on the vertices on S' are not all equal is equivalent to the condition that $n(S) = n(S')$. Hence $\{[\vec{u}_S]\}$ satisfies C_2. So the limit in $M[V]$ of $\vec{x}(t)$ as $t \to 0^+$, which equals $(\vec{x}, \{[\vec{u}_S]; \vec{x}|_S \in \Delta_S\})$, lies in $\mathscr{C}$.

Simple elaboration on this basic example proves that all points in $\mathscr{C}$ can be obtained this way. This shows $M[V] \supset \mathscr{C}$. We leave the reverse inclusion to the reader. One needs to show that a limit point in $M[V]$ is the limit of a curve $\vec{x}(t)$ as above using the compactness of the unit sphere bundle in $T(M)$.

5.3. Stratification of $M[V]$. Having shown that $M[V] = \mathscr{C}$, we can now decompose it into a disjoint union of open strata,

$$M[V] = \bigcup_{\mathscr{S}} M(\mathscr{S})^0. \tag{5.69}$$

Here $\mathscr{S}$ is a collection of subsets of $\underline{V}$, each subset containing two or more elements, which are nested: if sets S_1, S_2 belong to $\mathscr{S}$, they are either disjoint or else one contains the other.

The open strata $M(\mathscr{S})^0$ consists of the elements $(\vec{x}, \{\vec{x}_{B,S}\})$ of $M[V]$ satisfying the following conditions:

(i) $\vec{x}|_S \in \Delta_S$ exactly when $S \subset S'$ for some $S' \in \mathscr{S}$.

(ii) When S is the smallest set in $\mathscr{S}$ containing S', $[\vec{u}_{S'}] = [\vec{u}_S|_{S'}]$.

(iii) If $S_1, S_2 \in \mathscr{S}$ and $S_1 \subset S_2$, then $\vec{u}_{S_2}|_{S_1}$ is a constant map.

Conditions (ii) and (iii) together imply that the screens $\{[\vec{u}_S]; S \in \mathscr{S}\}$ are independent and determine the remaining screens.

S1 and S2 below should now be clear; S3 and S4 follow from our description of the manifold with corner structure on $M[V]$ in the next subsection.

S1: $M(\mathscr{S})^0$ is a smooth (noncompact) manifold of codimension $|\mathscr{S}|$ in $M[V]$, i.e., of dimension $3V - |\mathscr{S}|$.

S2: The closed strata $M(\mathscr{S})$, the closure of $M(\mathscr{S})^0$, equals $\bigcup_{\mathscr{T} \supseteq \mathscr{S}} M(\mathscr{T})^0$.

S3: The codimension-k boundary to $M[V]$ is the union of the codimension-k closed strata,

$$\partial_k M[V] = \bigcup_{\mathscr{S};|\mathscr{S}|=k} M(\mathscr{S}). \tag{5.70}$$

S4: $\partial_k M[V]\backslash\partial_{k+1}M[V]$ is the open set in $\partial_k M[V]$ given by $\bigcup_{\mathscr{S};|\mathscr{S}|=k} M(\mathscr{S})^0$.

For the codimension-1 strata needed in the next section, $\mathscr{S}$ consists of a single set S with $|S| > 1$. Then $M(\mathscr{S})^0$ is the set of pairs $(\vec{x}, \{[\vec{u}_S]\})$ for which $x_i = x_j$ if and only if $i, j \in S$ and the components of $\vec{u}_s$ are distinct and sum to zero.

5.4. Coordinates on $M[V]$. Let $c^{(0)} = (\vec{x}^{(0)}, \{[\vec{u}_S^{(0)}]; S \in \mathscr{S}\})$ be a point in $M[V]$ belonging to the open strata $M(\mathscr{S})^0$. We now define coordinates on $M[V]$ in a neighborhood of $c^{(0)}$. The definition will make use of a choice of $g \in \mathrm{Met}$. The collections of all coordinate systems as $c^{(0)}$ varies defines the manifold with corner structure of $M[V]$. This structure is independent of the choice of g. Having fixed g, we may choose $\vec{u}_S^{(0)}$ to be the unique representation of its screen with norm 1 and satisfying $\sum_{i\in S} u_{S,i}^{(0)} = 0$. Here $u_{S,i}^{(0)}$ is the value of $\vec{u}_S^{(0)}$ at the point $i \in S$.

Define a map $\psi\colon M(\mathscr{S})^0 \times [\mathbb{R}_{\geq 0}]^{\mathscr{S}} \to M^V$ by

$$\begin{aligned}
\psi(c, \vec{t}) &= (\underline{x}_1(c, t), \cdots, \underline{x}_V(c, t)), \\
\underline{x}_i(c, t) &= \exp_{x_i}\Big(\sum_{\substack{S\in \mathscr{S};\\ i\in S}} \tilde{t}_S u_{S,i}\Big), \\
\tilde{t}_S &= \prod_{\substack{S'\in\mathscr{S};\\ S'\supset S}} t_{S'},
\end{aligned} \tag{5.71}$$

where $c = (\vec{x}, \{[\vec{u}_S]; S \in \mathscr{S}\})$ and $\sum_{i\in S} u_{S,i} = 0$, $\|\vec{u}_S\| = 1$ for $S \in \mathscr{S}$.

Lemma. *There exist an open neighborhood U of $c^{(0)}$ in $M(\mathscr{S})^0$ and an open neighborhood W of $\vec{0}$ in $[\mathbb{R}_{\geq 0}]^{\mathscr{S}}$ so that the restriction $\psi_0 = \psi|_{U\times(W\backslash\partial W)}$ maps into M_0^V and is a diffeomorphism onto its image.*

Remark. It makes sense to claim that ψ_0 is a diffeomorphism since both $M(\mathscr{S})^0$ and $\mathbb{R}_{\geq 0}^{\mathscr{S}}\backslash\partial\mathbb{R}_{\geq 0}^{\mathscr{S}}$ are smooth manifolds (without corners).

Proof. By the inverse function theorem, it suffices to show that U and W may be chosen so that the following hold:

(i) ψ_0 maps into M_0^V.

(ii) The derivative of ψ_0 is injective.

(iii) ψ_0 is injective.

We need only consider the case where $\underline{V} \in \mathcal{S}$. During the proof we will identify a screen $[\vec{u}_S]$ at a point $\vec{x} = (x, \cdots, x)$ in the total diagonal in M^V with its preferred representative $\vec{u}_S$ of norm 1 satisfying $\sum_{i \in S} u_{S,i} = 0$. Recall that we set $u_{S,i} = 0$ for $i \notin S$ so that we may view $\vec{u}_S$ as an element of $TM^V_{\vec{x}}$. Let $\langle \cdot , \cdot \rangle$ denote the inner product on TM^V.

Proof of (i). By the tubular neighborhood theorem, it suffices to show that, for suitably small U and W,

$$\sum_{S \in \mathcal{S}} \tilde{t}_S u_{S,i} \neq \sum_{S \in \mathcal{S}} \tilde{t}_S u_{S,j} \tag{5.72}$$

for $i \neq j$, $(\vec{x}, \{\vec{u}_S\}) \in U$, and $\vec{t} \in W$. Let S_0 be the smallest set in $\mathcal{S}$ containing i and j. Since $u_{S,i} - u_{S,j} = 0$ for S not a subset of S_0, the difference of the two sides of (5.72) is

$$\tilde{t}_{S_0}(u_{S_0,i} - u_{S_0,j}) + \sum_{S \subsetneq S_0} \tilde{t}_S(u_{S,i} - u_{S,j}). \tag{5.73}$$

Note that for $S \subsetneq S_0$, $\tilde{t}_S$ equals $t_{S_0}\tilde{t}_S$ times a product of some other t's. Also $u_{S_0,i} - u_{S_0,j} \neq 0$. Hence, one can choose U and W small enough so that $|\tilde{t}_S(u_{S,i} - u_{S,j})|$ is much smaller than $|\tilde{t}_{S_0}(u_{S_0,i} - u_{S_0,j})|$, and therefore (5.73) is nonzero.

Proof of (ii). Using the tubular neighborhood theorem again as well as the fact that the map $\vec{t} \mapsto \vec{\tilde{t}}$ from $\mathbb{R}_+^{\mathcal{S}}$ to $\mathbb{R}_+^{\mathcal{S}}$ is a diffeomorphism, it suffices to show that the map $(\{\vec{u}_S\}, \vec{\tilde{t}}) \mapsto \sum_{S \in \mathcal{S}} \tilde{t}_S u_S$ is injective at the tangent space level. The derivative of this map in the direction of $(\{\delta\vec{u}_S\}, \delta\vec{\tilde{t}})$ is

$$\sum_{S \in \mathcal{S}} \vec{u}_S \delta\vec{t}_S + \tilde{t}_S \delta\vec{u}_S. \tag{5.74}$$

The fact that $\{\vec{u}_S\}$ is an orthonormal set of vectors in TM^V implies that $\langle \vec{u}_S, \delta\vec{u}_{S'} \rangle = 0$ for any $S, S' \in \mathcal{S}$, and that $\langle \delta\vec{u}_S, \delta\vec{u}_{S'} \rangle = 0$ for $S \neq S'$. Hence, all the individual terms in (5.74) are orthogonal. Therefore (5.74) is zero only if $\delta\vec{\tilde{t}}$ and the $\delta\vec{u}_S$ all vanish.

Proof of (iii). Using the tubular neighborhood theorem once more, it suffices to show that if U and W are suitably small, and $((\vec{x}, \{\vec{U}_S\}), \vec{t})$ and $((\vec{x}, \{\vec{u}_S\}), \vec{T})$ are points in $U \times (W \backslash \partial W)$ projecting to the same point $\vec{x} \in TM^V$, then $\sum_{S \in \mathcal{S}} \tilde{t}_S u_S$ equals $\sum_{S \in \mathcal{S}} \tilde{T}_S U_S$ only when $\tilde{t}_S = \tilde{T}_S$ and $u_S = U_S$ for all $S \in \mathcal{S}$. This follows because u_S and U_S have norm 1 and $\langle u_S, U_{S'} \rangle = 0$ for $S \neq S'$.

Theorem. *The map* ψ_0 *of the previous lemma extends continuously to a map* $\psi_B : U \times W \mapsto \mathrm{Im}(\psi_B) \subset M[V]$ *so that the following hold*:

T1. $\psi_B(c, \vec{t}) \in M(\mathscr{S}')^0$ *where* $\mathscr{S}' \equiv \{S \in \mathscr{S}\,;\, t_S = 0\}$.

T2. $\psi_B(c, \vec{0}) = c$.

T3. ψ_B *is a homeomorphism.*

T4. *The set of maps* ψ_B *as* c *varies over* $M[V]$ *is a system of coordinates on* $M[V]$ *giving it a structure of a manifold with corners.*

T5. *The manifold with corner structure on* $M[V]$ *is independent of the choice of metric* g.

T6. $M(\mathscr{S})^0$ *is an open subset of the smooth part of the codimension* $|\mathscr{S}|$ *boundary of* $M[V]$.

T7. *The inclusion map of* $M[V]$ *in* $\mathscr{B}$ *is smooth.*

Outline of Proof. Choose $(c, \vec{t}) \in U \times W$, $c = (\vec{x}, \{[\vec{u}_S]\,;\, S \in \mathscr{S}\})$ and let $\mathscr{S}' = \{S \in \mathscr{S}'\,;\, t_s = 0\}$ as above.

Let $\vec{\tau}: [0, \infty) \to M^V$ be the smooth curve

$$\vec{\tau}(\varepsilon) = \psi(c, \vec{t}_\varepsilon), \tag{5.75}$$

where $\vec{t}_\varepsilon \in \mathbb{R}^{\mathscr{S}}_{\geq 0}$ is given by

$$(t_\varepsilon)_S = \begin{cases} t_s & \text{for } S \notin \mathscr{S}', \\ \varepsilon & \text{for } S \in \mathscr{S}'. \end{cases} \tag{5.76}$$

Let $\underline{\vec{x}} = \vec{\tau}(0) = \psi(c, \vec{t})$. Observe that, when ε is a small positive number, $\tau(\varepsilon)$ equals $\psi_0(c, \vec{t}_\varepsilon) \in M_0^V$. Therefore, if ψ_B exists, it must equal

$$\psi_B(c, \vec{t}) = (\underline{\vec{x}}, \{[\underline{\vec{u}}_S]\,;\, (\underline{\vec{x}})|_S \in \Delta_S\}), \tag{5.77.1}$$

where

(5.77.2)

$$(\underline{\vec{x}}|_S, [\underline{\vec{u}}_S]) = \lim_{\varepsilon \to 0^+} \pi_{0,S}(\vec{\tau}(\varepsilon)) \quad \text{for } S \subset \underline{V},\ |S| \geq 2,\ (\underline{\vec{x}})|_S \in \Delta_S.$$

The limit in (5.77.2) can be calculated in terms of the Taylor series of $\tau(\varepsilon)$ in the manner introduced in §5.2. Write $\tau_i(\varepsilon) = \exp_{x_i}(w_i(\varepsilon))$, where

$$\begin{aligned} w_i(\varepsilon) &= \sum_{\substack{S \in \mathscr{S};\\ i \in S}} \varepsilon^{m(S)} \vec{t}'_S u_{S,i}, \\ m(S) &= |\{S' \in \mathscr{S}\,;\, S' \supset S,\ t_{S'} = 0\}| = |\{S' \in \mathscr{S}'\,;\, S' \supset S\}|, \\ \vec{t}'_S &= \prod_{\substack{S' \in \mathscr{S},\, t_{S'} \neq 0;\\ S' \supseteq S}} t_{S'}. \end{aligned} \tag{5.78}$$

Let $v_i(n)$ be the coefficient of ε^n in $w_i(\varepsilon)$. Then

$$w_i(\varepsilon) = \sum_{n=0}^{|\mathscr{S}|} v_i(n)\varepsilon^n, \tag{5.79.1}$$

$$v_i(n) = \sum_{\substack{S' \in \mathscr{S};\, i \in S';\\ m(S')=n}} \tilde{t}'_{S'} u_{S',i}. \tag{5.79.2}$$

Observe that $\underline{x}_i = \exp_{x_i}(v_i(0))$ and also that when $n = 0$ the terms in the sum (5.79.2) have $m(S') = 0$; so $\tilde{t}'_{S'} = \tilde{t}_{S'}$. A little thought suffices to verify that

$$[\underline{x}_i = \underline{x}_j] \Leftrightarrow [x_i = x_j \text{ and } v_i(0) = v_j(0)] \Leftrightarrow [i, j \in S' \text{ for some } S' \in \mathscr{S}'].$$

So the limit in (5.77.2) must be calculated for $S \subset \underline{V}$ with $|S| \geq 2$ and $S \subset S_0$ for some $S_0 \in \mathscr{S}'$. Fix such an S until further notice.

Let $z = \underline{x}_i$ for $i \in S$, and $F_Z\colon TM_{x_i} \to TM_z$ be the vector space isomorphism

$$F_z(w) = \left.\frac{d}{d\kappa}\right|_{\kappa=0} \exp_{x_i}(v_i(0) + \kappa w) \quad \text{for } w \in TM_{x_i}. \tag{5.80}$$

(I.e., $F_z(w)$ is obtained from w by transport using the Jacobi equation, the geodesic deviation equation). Then

$$\tau_i(\varepsilon) = G\left(\sum_{n=1}^{|\mathscr{S}|} F_z(v_i(n))\varepsilon^n\right), \tag{5.81.1}$$

where

$$G(a) = \exp_{x_i}(v_i(0) + (F_z)^{-1}(a)) \quad \text{for } a \in TM_z. \tag{5.81.2}$$

The argument of G in (5.81.1) is the version in the present context of the right-hand side of (5.67). Using the map G simply provides an invariant way of identifying points near z with points in TM_z. If we choose g to be flat near x_i and work in flat coordinates, G and F_z become trivial.

Set $n(S) = \min\{n;\, v_i(n) \neq v_j(n) \text{ for some } i, j \in S\}$ as in the paragraph below (5.67). Using the facts that

$$S' \subset S \Rightarrow m(S') \geq m(S)$$

and

$$[S' \in \mathscr{S},\ i, j \in S \cap S',\ m(S') < m(S)] \Rightarrow u_{S',i} = u_{S',j}$$

and the technique used to prove point (iii) in the previous lemma, it is not hard to show that $n(S) = m(S)$. Hence, as in the sentence after (5.68),

$$\underline{\vec{u}}_{S,i} = F_z(v_i(n(S))) = \sum_{\substack{S' \in \mathcal{S};\, i \in S' \\ m(S')=m(S)}} \vec{t}_{S'} F_z(u_{S',i}) \tag{5.82}$$

for $i \in S$. Note that the sets S' in the sum in (5.82) need not necessarily be contained in or contain S.

Specializing (5.82) to the case where $\vec{t} = 0$, we obtain $\underline{\vec{u}}_S = \vec{u}_{S_1}|_S$, where S_1 is the smallest element of $\mathcal{S}$ containing S. This verifies T2.

Using (5.82) we find: if $S_1, S_2 \subset S_0 \in \mathcal{S}'$, $|S_1| \geq 2$ and $S_2 \subsetneqq S_1$, then

$$[\vec{u}_{S_1}|_{S_2} \text{ is constant}] \Leftrightarrow [\exists S_3 \in \mathcal{S}';\, S_2 \subset S_3 \subsetneqq S_1].$$

This statement is equivalent to T1.

The verification of T3, that the extension ψ_B of ψ_0 defined by (5.77.1) and (5.82) is a homeomorphism, is an exercise in point set topology.

To prove T4 and T5 it is necessary to show that the overlap map between coordinate charts ψ_B, ψ_B', associated to choices of $c, c' \in M[V]$ and $g, g' \in \mathrm{Met}$, is a diffeomorphism from one manifold with corners (an open subset of $U \times W$) to another (an open subset of $U' \times W'$). We will not carry out this very tedious exercise here. It can be derived more conceptually by using the map $\psi_{B,a} = \phi \circ \psi_B \colon U \times W \to M_a[V]$ which is given explicitly by (5.82) together with $\psi_{B,a}(c, \vec{t}) = (\underline{\vec{x}}, \{[\underline{\vec{u}}_S]_a\})$, where $[\underline{\vec{u}}_S]_a$ is the orbit of $\underline{\vec{u}}_S$ under the combined action of translation by T_zM and multiplication by $\mathbb{R}\backslash\{0\}$ (rather than $\mathbb{R}_+$). One can check that $\psi_{B,a}$ extends to a coordinate chart for $M_a[V]$ (by allowing the t_S's to be negative). The overlap maps of the ψ_B's are the restriction to nonnegative t_S's of the overlap maps for the $\psi_{B,a}$'s and hence are smooth.

Finally, T6 and T7 follow by inspection.

6. Main theorem

Our first basic result in [2] was that the integrals defining $I(\mathbf{G})$ are convergent despite the singularities near the union of all the diagonals in M^V. In fact, we prove a strong version of this in [2] using power-counting techniques of physics. We showed that the integral $\int_{M^V} \mathrm{Tr}^{(v)}(\mathcal{F}(\underline{\mathbf{G}})\Psi)$ converges for any smooth $\Psi \in \Omega^*(M^V, A_V^*)$ and any $\mathbf{G}$ (not necessarily trivalent). In the language of quantum field theory, this says that Chern-Simons perturbation theory is finite.

The main result of this paper is to prove the conjecture made in [2] that the dependence of I_l^{conn} on the arbitrary choice of g could be cancelled by subtracting a local counterterm which is an appropriate multiple of the "gravitational" Chern-Simons invariant $CS_{\text{grav}}(g, s)$ of the metric connection on M, defined using a homotopy framing s of TM (see [2]). Stated another way, we have our main theorem.

Theorem. *There is a constant β_l depending only on l and the bi-invariant inner product $\langle\ ,\ \rangle_{\text{Lie}(G)}$ on* Lie(G) *so that the quantity*

$$\widehat{I}_l^{\text{conn}}(M, A_0, s) \equiv I_l^{\text{conn}}(M, A_0, g) - \beta_l CS_{\text{grav}}(g, s)$$

is independent of g. $\widehat{I}_l^{\text{conn}}$ is therefore a topological invariant depending on the choice of manifold M, homotopy framing s, and flat connection A_0.

When l is odd, $\beta_l = 0$.

Remark 1: The naturality of our construction implies that the values of $\widehat{I}_l^{\text{conn}}$ agree for two choices of (M, A_0, s) which are related by a principal bundle automorphism. (Although the automorphisms must be differentiable, we use the term topological invariant since it is more standard in this context.)

The proof of the theorem has three steps. First, we shall rewrite I_l as a push-forward by integration over $M[V]$ of a closed form on $M[V] \times \text{Met}$ constructed from $\widetilde{L}$. Next we shall apply Stokes theorem to write the anomaly $d_{\text{Met}} I_l$ as an integral over the boundary of $M[V]$. Then we shall use the explicit descriptions of the propagator singularities (i.e., $\widetilde{L}|_{\partial B_2 \times \text{Met}}$) and of $\partial M[V]$ to calculate $d_{\text{Met}} I_l$. This result will imply that

$$d_{\text{Met}} I_l^{\text{conn}} = \beta_l CS_{\text{grav}}(g, s)$$

as desired.

Step 1: Rewriting I_l. Shortly we will define the total propagator $\widetilde{L}_{C,\text{tot}}$ on the compactification $M[V] \times \text{Met}$. It belongs to $\Omega^*(M[V] \times \text{Met}, \widetilde{A}_V^*)$. $\widetilde{A}_V^*$ stands for the bundle A_V^* pulled back to either $M^V \times \text{Met}$ or, in this case, to $M[V] \times \text{Met}$. $\widetilde{L}_{C,\text{tot}}$ is characterized by the facts that it is smooth on all of $M[V]$ and that it agrees with $\widetilde{L}_{\text{tot}}$ (the analog of L_{tot} defined using the extended propagator) on $M_0^V \times \text{Met}$.

Having defined $M[V]$ and $\widetilde{L}_{C,\text{tot}}$, we may rewrite I_l in terms of them:

$$I_l = \int_{M[V]} \text{Tr}^{(V)}(\widetilde{L}_{C,\text{tot}}^I). \tag{6.83}$$

The operator $\mathrm{Tr}^{(V)}$ is the same as the operator $\mathrm{Tr}^{(V)}$ defined previously but now maps $\Omega^*(M[V] \times \mathrm{Met}, \widetilde{A}_V^*)$ to $\Omega^*(M[V] \times \mathrm{Met})$.

The integral in (6.83) agrees with $\int_{M^V} \mathrm{Tr}^{(V)}(L_{\mathrm{tot}}^I)$. Since the integrand has degree $3V = \dim(M^V)$ as a differential form, the integral picks out the piece of $\widetilde{L}_{\mathrm{tot}}^I$ of degree 0 in the Met directions. This is precisely L_{tot}^I. Thus (6.83) agrees with the previous definition of I_l.

Now we define $\widetilde{L}_{C,\mathrm{tot}}$. As one would expect, it is a double sum

$$\widetilde{L}_{C,\mathrm{tot}} = \sum_{i,j=1}^{V} \widetilde{L}_{C,\{i,j\}} \tag{6.84}$$

of pieces $\widetilde{L}_{C,\{i,j\}} \in \Omega^2(M[V] \times \mathrm{Met}, A_V^2)$. $\widetilde{L}_{C,\{i,j\}}$ smoothly extends $\widetilde{L}_{ab}(x_i, x_j) j_{(i)}^{\cdot a} j_{(j)}^{\cdot b}|_{M_0^V \times \mathrm{Met}}$ to all of $M[V] \times \mathrm{Met}$, as follows.

Let $\pi_{B,\{i,j\}} : M[V] \to \mathrm{Bl}(M^{(\{i,j\}}, \Delta_{\{i,j\}})$ be the map

$$\pi_{B,\{i,j\}}((\vec{x}, \{\vec{x}_{B,S}; S \subset \underline{V}, |S| > 2\})) = \vec{x}_{B,\{i,j\}} \tag{6.85}$$

and $f_{B,\{i\}} : M[V] \to M^{\{i\}}$ be the map

$$f_{B,\{i\}}((\vec{x}, \{\vec{x}_{B,S}; S \subset \underline{V}, |S| > 2\})) = x_i. \tag{6.86}$$

The trivial cross product of these maps with Met will be denoted by $\tilde{\pi}_{B,\{i,j\}}$ and $\tilde{f}_{B,\{i\}}$.

For $i \neq j$, $\widetilde{L}_{C,\{i,j\}}$ is given by

$$\widetilde{L}_{C,\{i,j\}} = (\pi_{B,\{i,j\}})^*(\widetilde{L}_{Bs,\{i,j\}}) = (\widetilde{L}_B)_{ab}(\vec{x}_{B,\{i,j\}}) j_{(i)}^{\cdot a} j_{(j)}^{\cdot b}. \tag{6.87}$$

Here $\widetilde{L}_{Bs,\{i,j\}}$ is a copy of the "extended superpropagator" $\widetilde{L}_{Bs} = s(\widetilde{L}_B)$, but for the vertices i, j rather than 1, 2. For $i = j$, the appropriate definition is

$$\widetilde{L}_{C,\{i,i\}} = (f_{B,\{i\}})^*(\tilde{\rho}_{s,\{i\}}) = \tilde{\rho}_{ab}(x_i, x_i) j_{(i)}^{\cdot a} j_{(i)}^{\cdot b}. \tag{6.88}$$

$\tilde{\rho}_{s,\{i\}}$ here is a copy of $\tilde{\rho}_s = s(\tilde{\rho})$ belonging to $\Omega^*(M^{\{i\}} \times \mathrm{Met}, \Lambda^2(\tilde{\mathbf{g}}_1 \otimes \tilde{\mathbf{g}}_2))$.

For notational convenience in (6.87), (6.88), and below, we have not written the argument g explicitly.

Remark 2: Stokes Theorem. Let d_{Met} and $d_{M[V]}$ be the exterior derivative operators. Since $\widetilde{L}_{C,\mathrm{tot}}$ is covariantly closed, the integrand in (6.83) is closed. Hence,

$$
\begin{aligned}
d_{\mathrm{Met}} I_l &= c_{V,I} \int_{M[V]} d_{\mathrm{Met}} \mathrm{Tr}^{(V)}(\widetilde{L}^I_{C,\mathrm{tot}}) \\
&= c_{V,I} \int_{M[V]} -d_{M[V]} \mathrm{Tr}^{(V)}(\widetilde{L}^I_{C,\mathrm{tot}}) \\
&= -c_{V,I} \int_{\partial M[V]} \mathrm{Tr}^{(V)}(\widetilde{L}^I_{C,\mathrm{tot}}).
\end{aligned} \tag{6.89}
$$

Remark 3: Calculation of the anomaly. Because $\widetilde{L}_{C,\mathrm{tot}}$ is smooth,, we are free to replace $\partial M[V]$ in (6.89) by the open dense subset $\partial M[V] \backslash \partial_2 M[V]$. The latter is the disjoint union of the codimension-1 open strata:

$$
\partial M[V] \backslash \partial_2 M[V] = \bigcup_{\underline{V}' \subset \underline{V},\, |\underline{V}'| \geq 2} M(\{\underline{V}'\})^0 \subset^{\mathrm{open}}_{\mathrm{dense}} \partial M[V]. \tag{6.90}
$$

Furthermore, two different choices of $\underline{V}'$ which differ only by a permutation of $\underline{V}$ give equal contributions. Therefore, by including a combinatorial factor, we may restrict to the standard choices $\underline{V}' = \{1, \cdots, V'\}$ $(2 \leq V' \leq V)$. Thus, we obtain

$$
d_{\mathrm{Met}} I_l = -c_{V,I} \sum_{V'=2}^{V} \binom{V}{V'} \int_{M(\underline{V}')^0} \mathrm{Tr}^{(V)}(\widetilde{L}^I_{C,\mathrm{tot}}). \tag{6.91}
$$

The term in (6.91) with a given V' is the contribution to the anomaly from the regions where V' points coincide. It will be useful to introduce names $V'' \equiv V - V'$ and $\underline{V}'' \equiv \{V' + 1, \cdots, V\}$ for the number of and the label set for the points not coinciding.

Recall from §5 that $M(\{\underline{V}'\})^0$ equals the set of $(\vec{x}, \{[\vec{u}_{\underline{V}'}]\})$ where

(i) $\vec{x} = (x_1, \cdots, x_V)$ is an element of M^V with x_1 through $x_{V'}$ all equal to some z in M_z (which is just a disjoint copy of M labeled by z) and all pairs x_i, x_j distinct otherwise; and

(ii) $[\vec{u}_{\underline{V}'}]$ is an element of the fiber of the sphere bundle $S([TM_z]^{\underline{V}'}/TM_z)$ at z represented by a vector $(u_1, \cdots, u_{V'}) \in [T_z M_z]^{\underline{V}'}$ with no two components equal.

For $i \neq j$, we also have

(6.92)
$$\vec{x}_{B,\{i,j\}} = \pi_{B,\{i,j\}}(\vec{x}, \{[\vec{u}]\}) = \begin{cases} ((z, z), [(u_i, u_j)]) \in \partial \mathrm{Bl}(M^{(\{i,j\}}, \Delta_{\{i,j\}}), & i, j \in \underline{V}', \\ (x_i, x_j) \in \mathrm{Bl}(M^{\{i,j\}}, \Delta_{\{i,j\}}) \backslash \partial \mathrm{Bl}(M^{\{i,j\}}, \Delta_{\{i,j\}}) & \text{otherwise.} \end{cases}$$

As a particular case of the bottom line, $\pi_{B,\{i,j\}}(\vec{x}, \{[\vec{u}_{\underline{V}'}]\}) = (z, x_j)$ for $i \leq V' < j$; and similarly for i and j reversed.

This description of $\pi_{B,\{i,j\}}$ on $M(\{\underline{V}'\})^0$ allows us to write

(6.93)
$$\widetilde{L}_{C,\{i,j\}} = \begin{cases} \tilde{\rho}(x_i, x_i) j^a_{(i)} j^b_{(i)}, & i = j > V', \\ \tilde{\rho}(z, z) j^a_{(i)} j^b_{(i)}, & i = j \leq V', \\ \widetilde{L}_{ab}(x_i, x_j) j^a_{(i)} j^b_{(j)}, & i, j > V', \\ & i \neq j, \\ \widetilde{L}_{ab}(z, x_j) j^a_{(i)} j^b_{(j)}, & i \leq V',\ j > V', \\ \widetilde{L}_{ab}(x_i, z) j^a_{(i)} j^b_{(j)}, & i > V',\ j \leq V', \\ [\tilde{\lambda}(z, [(u_i, u_j)])\delta_{ab} + \tilde{\rho}_{ab}(z, z)] j^a_{(i)} j^b_{(j)}, & i, j \leq V', i \neq j. \end{cases}$$

Thus we may decompose $\widetilde{L}_{C,\mathrm{tot}}$ into terms coming from the explicit propagator singularity and remaining "regular" terms,

(6.94.1)
$$\widetilde{L}_{C,\mathrm{tot}} = \widetilde{L}_{\mathrm{sing},V'} + \widetilde{L}_{\mathrm{reg},V''},$$

(6.94.2)
$$\widetilde{L}_{\mathrm{sing},V'} = \sum_{\substack{i,j \leq V' \\ i \neq j}} \tilde{\lambda}(z, [(u_i, u_j)])[j^a_{(i)} \wedge j^a_{(j)}],$$

(6.94.3)
$$\begin{aligned} \widetilde{L}_{\mathrm{reg},V''} = \tilde{\rho}_{ab}(z, z) J^a \wedge J^b &+ \sum_{i>V'} \tilde{\rho}_{ab}(x_i, x_i) j^a_{(i)} \wedge j^b_{(i)} \\ &+ \sum_{j>V'} \widetilde{L}_{ab}(z, x_j) J^a \wedge j^b_{(j)} \\ &+ \sum_{i>V'} \widetilde{L}_{ab}(x_i, z) j^a_{(i)} \wedge J^b \\ &+ \sum_{\substack{i,j>V' \\ i \neq j}} \widetilde{L}_{ab}(x_i, x_j) j^a_{(i)} \wedge j^b_{(j)}, \end{aligned}$$

where

$$J^a = \sum_{1 \le i \le V'} j^a_{(i)}. \tag{6.94.4}$$

The important properties of (6.94) which we need are the following.

P1. $\widetilde{L}_{\mathrm{reg}, V''}$ only depends on (z, J) and the $(x_i, j_{(i)})$ for $i > V'$.

P2. $\widetilde{L}_{\mathrm{sing}, V'}$ only depends on the $(x_i, j_{(i)})$ for $i \le V'$, and on $[\vec{u}_{\underline{V}'}]$.

P3. Each term in the sum (6.94.2) defining $\widetilde{L}_{\mathrm{sing}, V'}$ factors into a "group theory piece" $(j^a_{(i)} \wedge j^a_{(j)})$ times a "manifold piece" $(\tilde{\lambda}(z, [u_i, u_j]))$.

P4. $\widetilde{L}_{\mathrm{sing}, V''}$ is invariant under diagonal gauge transformations, that is, gauge transformations that acts the same on all factors $\tilde{\mathbf{g}}_1, \cdots, \tilde{\mathbf{g}}_{V'}$.

P4 follows from the invariance of the Lie algebra metric.

Substituting the first line of (6.94) into (6.91) and expanding by the binomial theorem, one finds

$$\begin{aligned} d_{\mathrm{Met}} I_l = &- \sum_{V'=2}^{V} \sum_{I'=0}^{I} c_{V', I'} c_{V'', I''} \\ &\times \int_{S(TM_z^{\underline{V}'}/TM_z) \times M^{\underline{V}''}} \mathrm{Tr}^{(V)}([\widetilde{L}_{\mathrm{sing}, V'}]^{I'} \wedge [\widetilde{L}_{\mathrm{reg}, V''}]^{I''}), \end{aligned} \tag{6.95}$$

where $I'' = I - I'$. The domain of integration indicated gives the same result as $M(\underline{V}')^0$ which is an open dense subset.

The next step consists of breaking the integral up into three parts: (1) an integral over $(x_{V'+1}, \cdots, x_V)$ in $M^{\underline{V}''}$, together with the Lie algebra traces for $j_{(V'+1)}, \cdots, j_{(V)}$ in $\tilde{\mathbf{g}}^{\underline{V}''} = \tilde{\mathbf{g}}_{V'+1} \oplus \cdots \oplus \tilde{\mathbf{g}}_V$; (2) an integral over $[\vec{u}]$ in $S(TM_z^{\underline{V}'}/TM_z)|_z$ for fixed z, together with the contractions over the nondiagonal directions in $\tilde{\mathbf{g}}^{\underline{V}'} = \tilde{\mathbf{g}}_1 \oplus \cdots \oplus \tilde{\mathbf{g}}_{V'}$; and finally (3) an integral over z in M_z together with contractions for J which belongs to the diagonal directions $\tilde{\mathbf{g}}_J \subset \tilde{\mathbf{g}}^{\underline{V}'}$.

Before proceding we explain the phrase "contraction over the diagonal directions". Write $\tilde{\mathbf{g}}^{\underline{V}''} = \mathbf{h} \oplus \tilde{\mathbf{g}}_J$, where

$$\mathbf{h} = \left\{ (j_1, \cdots, j_{\underline{V}'}) \in \tilde{\mathbf{g}}^{\underline{V}'}; \sum_{i=0}^{V'} j_{(i)} = 0 \right\}, \tag{6.96}$$

and $\tilde{\mathbf{g}}_J$ is its orthogonal. The subspace $\tilde{\mathbf{g}}_J$ is the space of diagonal directions, that is,

$$\tilde{\mathbf{g}}_J = \{(j_{(1)}, \cdots, j_{(V')}) \in \tilde{\mathbf{g}}^{V'}; j_{(r)} = j_{(s)}, \ r, s \in \underline{V}'\}. \tag{6.97}$$

So

$$\Lambda^{2I'}(\tilde{\mathbf{g}}^{\underline{V'}}) = \sum_r \Lambda^{2I'-r}(\mathbf{h}) \otimes \Lambda^r(\tilde{\mathbf{g}}_J). \tag{6.98}$$

Contraction over the nondiagonal directions means interpreting $\eta \in \Lambda^{2I'}(\tilde{\mathbf{g}}^{\underline{V'}})$ as a linear function acting on $\omega \in \Omega^E(\tilde{\mathbf{g}}_J)$ by wedging to get $\eta \wedge \omega \in \Lambda^{2I'+E}(\tilde{\mathbf{g}}^{\underline{V'}})$ and applying $\mathrm{Tr}^{(V')}$, giving a real number $\mathrm{Tr}^{(V')}(\eta \wedge \omega)$ (which vanishes unless $E = 3V' - 2I'$).

We may write

$$d_{\mathrm{Met}} I_l = \sum_{V'=2}^{V} \sum_{I'=0}^{I} \int_M \langle \overline{A}_{V',I'}, \overline{C}_{V'',I''} \rangle. \tag{6.99}$$

We now discuss the two pieces $\overline{A}_{V',I'}$ and $\overline{C}_{V'',I''}$ of this equation.

$\overline{C}_{V'',I''}$ is the result of pushing forward $[\widetilde{L}_{\mathrm{reg},V''}]^{I''}$, considered (by P1) as an element of $\Omega^{2I''}(M_z \times M^{\underline{V''}} \times \mathrm{Met},\ \Lambda^{2I''}(\tilde{\mathbf{g}}_J \times \tilde{\mathbf{g}}^{\underline{V''}}))$, by integration over $M^{\underline{V''}}$ and contraction on $\tilde{\mathbf{g}}^{\underline{V''}}$,[4]

(6.100)

$$\begin{aligned} &\overline{C}_{V'',I''}(z, J) \\ &\quad = c_{V'',I''} \int_{(x_{V'+1}, \cdots, x_V) \in M^{V''}} \mathrm{Tr}_{V'+1} \circ \cdots \circ \mathrm{Tr}_V([\widetilde{L}_{\mathrm{reg},V''}]^{I''}). \end{aligned}$$

Since the integration subtracts manifold form degree $3V''$, and the Lie algebra traces subtract Lie algebra form degree $3V''$, $\overline{C}_{V'',I''}$ belongs to $\Omega^E(M_z \times \mathrm{Met}, \Lambda^E(\tilde{\mathbf{g}}_J))$, where

$$E = 2I'' - 3V'' = 3V' - 2I'. \tag{6.101}$$

Similarly $\overline{A}_{V',I'}$ is the push-forward of $\widetilde{L}^{I'}_{\mathrm{sing},V'}$, considered (by P2) as an element of $\Omega^{2I'}(S(TM_z^{\underline{V'}}/TM_z) \times \mathrm{Met},\ \Lambda^{2I'}(\tilde{\mathbf{g}}^{\underline{V'}}))$, by integration over the fibers of $S(TM_z^{\underline{V'}}/TM_z) \to M_z$ and contractions in the nondiagonal directions, as explained above. Thus, for $\omega(z, J) \in \Lambda^*(\tilde{\mathbf{g}}_J)|_z$, we have

(6.102)

$$\begin{aligned} &\langle \overline{A}_{V',I'}(z, J), \omega(z, J) \rangle \\ &\quad = -c_{V',I'} \int_{[\vec{u}] \in S(TM_z^{\underline{V'}}/TM_z)} \mathrm{Tr}_1 \circ \cdots \circ \mathrm{Tr}_{V'}([\widetilde{L}_{\mathrm{sing},V'}]^{I'} \wedge \omega(z, J)). \end{aligned}$$

[4]In physical parlance, $\overline{C}_{V'',I''}$ is the untruncated Green's function with E external legs and at order $(I'' - E) - V'' + 1$ in perturbation theory, evaluated on the superspace diagonal (meaning that all external vertices and generalized polarization tensors agree).

The degree as a differential form of $\overline{A}_{V',I'}$ is

$$2I' - \dim(S(TM_z^{V'}/TM_z)|_z) = 2I' - (3V' - 4) = 4 - E, \tag{6.103}$$

whereas it pairs with Lie algebra forms of degree $3V' - 2I' = E$. $\overline{A}_{V',I'}$ is invariant under gauge transformations, as follows from invariance of $\widetilde{L}_{\text{sing},V'}$ (see P4) and the operators Tr_i for $i \leq V'$. Putting this together, $\overline{A}_{V',I'}$ belongs to $\Omega^{4-E}(M_z \times \mathrm{Met}, [\Lambda^E(\tilde{\mathbf{g}}_J)^\vee]^{\mathrm{invt}}$, where $W^\vee$ denotes the dual of a vector space W.

One can now expand each factor of $\widetilde{L}_{\text{sing},V'}$ appearing in (6.102) using (6.94.2) and rewrite the resulting sum as a sum over labeled graphs, as was done in arriving at (3.22). By property P3 above, the contribution of each graph factors into a manifold piece times a group theory piece. The result is

$$\overline{A}_{V',I'} = -c_{V',I'} \sum_{\underline{\mathbf{G}}'} A_{\text{mfld}}(\mathbf{G}') A_{\text{gp}}(\mathbf{G}'), \tag{6.104.1}$$

where

$$A_{\text{mfld}}(\mathbf{G}') \equiv \int_{\{[\vec{u}]\}} \prod_{e=1}^{I'} \tilde{\lambda}(z, [u_{i_e}, u_{j_e}]) \in \Omega^{4-E}(M_z \times \mathrm{Met}), \tag{6.104.2}$$

and

$$\begin{aligned} A_{\text{gp}}(\mathbf{G}') &\equiv \mathrm{Tr}_1 \circ \cdots \circ \mathrm{Tr}_{V'} \circ \prod_{e=1}^{I'} j^{.a}_{(i_e)} j^{.a}_{(i_e)} \in [\Lambda^E(\tilde{\mathbf{g}}_J)^\vee]^{\mathrm{invt}} \\ &\cong \Lambda^E(\mathrm{Lie}(G)^\vee)^{\mathrm{invt}}. \end{aligned} \tag{6.104.3}$$

The sum in (6.104.1) is over all labeled graphs $\underline{\mathbf{G}}'$ with V' vertices and I' edges which have no vertices of valency greater than 3 (and also no edges connecting a vertex to itself). E is the number of external edges the graphs have, i.e., the number of edge ends that need to be attached to any of the $\mathbf{G}'$ to make it a trivalent graph. These graphs have

$$l' = I' - V' + 1 \tag{6.105}$$

loops. Note that

$$V' = 2(l' - 1) + E \quad \text{and} \quad I' = 3(l' - 1) + E. \tag{6.106}$$

Also note that the graphs may be assumed connected since the integral (6.104.2) vanishes for $\mathbf{G}'$ disconnected. The vanishing follows because the integrand is annihilated by interior product by the nontrivial vector field which scales the u_i for i labeling one of the vertices in a connected component of $\mathbf{G}'$ (see [2]).

The expression on the right of (6.104.3) is an operator that when acting on $\Lambda^E(\tilde{\mathbf{g}}_J)$ produces a number, as in (6.102). Since it is gauge invariant and its explicit form does not depend on z, it may be viewed as an element of the fixed space $[\Lambda^E(\tilde{\mathbf{g}}_J)^\vee]^{\mathrm{invt}} \cong \Lambda^E(\mathrm{Lie}(G)^\vee)^{\mathrm{invt}}$. $A_{\mathrm{gp}}(\mathbf{G}')$ only depends on the (unlabeled) graph $\mathbf{G}'$, the group G, and the invariant metric $\langle\ ,\ \rangle_{\mathrm{Lie}(G)}$ on $\mathrm{Lie}(G)$. Its value remains unchanged if we replace G by its semisimple part G_{ss}, and $\langle\ ,\ \rangle_{\mathrm{Lie}(G)}$ by its restriction to G_{ss}). This follows because the structure constants (appearing in Tr_i) in the direction of the $U(1)$ factors of G vanish. With this replacement of G by G_{ss}, $A_{\mathrm{gp}}(\mathbf{G}')$ becomes an element of $\Lambda^E(\mathrm{Lie}(G_{\mathrm{ss}})^\vee)^{\mathrm{invt}}$, considered as a subspace of $\Lambda^E(\mathrm{Lie}(G)^\vee)^{\mathrm{invt}}$.

Note that $A_{\mathrm{mfld}}(\mathbf{G}')$ is a characteristic polynomial of $\widetilde{\Omega}$. In other words

$$A_{\mathrm{mfld}}(\mathbf{G}') = P_{\mathbf{G}'}(\widetilde{\Omega}), \tag{6.107}$$

where $P_{\mathbf{G}'}$ is an invariant symmetric tensor on $\mathrm{Lie}(SO(3))$. This follows because $\tilde{\lambda}(z, [u_i, u_j])$ equals a combination of vertical forms (along the directions of the fiber $S(TM_z^{V'}/TM_z)$ being integrated over) and the pullback of $\widetilde{\Omega}$ and because this combination is invariant under the $SO(TM_z)$ action on the u_i's and on $\widetilde{\Omega}$. Since $\tilde{\lambda}$ is universal, $P_{\mathbf{G}'}$ only depends on $\mathbf{G}'$.

Now we come to the heart of the proof. Up to now, our calculation of the anomaly would apply, with a little modification, to calculating gauge fixing anomalies in a wide class of theories; although the particular form for A_{mfld} and A_{gp} would be different. Now we will use those particular forms to prove that $\overline{A}_{V',I'}$ vanishes unless $E = 0$. To begin, $\overline{A}_{V',I'}$ must have nonnegative degree as a differential form, so $E \leq 4$. Next, since $\widetilde{\Omega}$ has degree 2, (6.107) implies that E must be even if $\overline{A}_{V',I'}$ is to be nonzero. This leaves only $E = 2$ or 4. Finally, those cases are handled because $\Lambda^E(\mathrm{Lie}(G_{\mathrm{ss}})^\vee)^{\mathrm{invt}}$ is isomorphic to the cohomology group $H^E(G_{\mathrm{ss}}; \mathbb{R})$. By semisimplicity, the latter group is trivial for $E = 2$ or 4.

When $E = 0$, the terms involving J in (6.94.3) for $\widetilde{L}_{\mathrm{reg}, V''}$ do not contribute to $\overline{C}_{V'',I''}$. $\overline{C}_{V'',I''}$ is also independent of z. In fact, changing labeling set from $\{V'+1, \cdots, V\}$ to $\{1, \cdots, V''\}$ on the right side of (6.100), we obtain the definition of one of the original perturbative invariants,

$$\overline{C}_{V',I'} = I_{l''} \quad \text{for } E = 0 \text{ and } l'' = I'' - V'' + 1. \tag{6.108}$$

Putting all of the above together, we have

$$d_{\text{Met}} I_l = \sum_{\substack{l'=2 \\ l''=l-l'}} A_{l'} I_{l''}, \tag{6.109.1}$$

$$A_{l'} \equiv \int_M \overline{A}_{2(l'-1),3(l'-1)} = -c_{V',I'} \sum_{\underline{\mathbf{G}'}} A_{\text{gp}}(\mathbf{G}') \int_M P_{\mathbf{G}'}(\widetilde{\Omega}), \tag{6.109.2}$$

where the sum is over labeled connected trivalent graphs with l' loops.

In (6.109), $P_{\mathbf{G}'}$ is an invariant tensor on $\text{Lie}(SO(3))$ of degree 2. This implies that it must be a multiple of the inner product. So

$$P_{\mathbf{G}'}(\widetilde{\Omega}) = (\alpha(\mathbf{G}')/8\pi)\langle\widetilde{\Omega}, \widetilde{\Omega}\rangle \tag{6.110}$$

for $\alpha(\mathbf{G}')$ a constant which depends only on $\mathbf{G}'$. But, with any choice of framing, the variation of the Chern-Simons action of the metric connection is given by

$$d_{\text{Met}} CS_{\text{grav}}(g, s) = \frac{1}{8\pi}\int_M 2\langle\Omega, \delta\Gamma\rangle = \frac{1}{8\pi}\int_M \langle\widetilde{\Omega}, \widetilde{\Omega}\rangle. \tag{6.111}$$

To obtain the last equality in (6.111), recall ((4.33)) that

$$\widetilde{\Omega} = \Omega + \delta\Gamma - \tfrac{1}{2}\nabla(g^{-1}\delta g) - \tfrac{1}{4}(g^{-1}\delta g)\wedge(g^{-1}\delta g). \tag{6.112}$$

The term involving $\nabla\delta g$ vanishes by integration by parts and the Bianchi identity.

Putting the results of the last paragraph into (6.109.2) yields

$$\begin{aligned} A_{l'} &= \beta_{l'}\, d_{\text{Met}} CS_{\text{grav}}(g, s), \\ \beta_{l'} &= -c_{V',I'} \sum_{\underline{\mathbf{G}'}} A_{\text{gp}}(\mathbf{G}')\alpha(\mathbf{G}'). \end{aligned} \tag{6.113}$$

$\beta_{l'}$ depends only on l' and on the metric on $\text{Lie}(G)$ (or even just its restriction to $\text{Lie}(G_{\text{ss}})$).

The desired result

$$d_{\text{Met}} I_{l'}^{\text{conn}} = A_{l'} = \beta_{l'}\, d_{\text{Met}} CS_{\text{grav}}(g, s)$$

follows from (6.109.1), (6.113), and the standard relation

$$(6.114)\qquad 1+\sum_{l=2}^{\infty} I_l k^{1-l} = \exp\left(\sum_{l'=2}^{\infty} I_{l'}^{\mathrm{conn}} k^{1-l'}\right) \in \mathbb{R}[[k^{-1}]]$$

between sums over all graphs and connected graphs.

The only thing left now to of complete the proof is to show that β_l vanishes for l odd. It suffices to show that the right-hand side of (6.104.2) vanishes when l' is odd. This follows by looking at the involution $[\vec{u}] \to [-\vec{u}]$ of the integration region $S(TM_z^{\underline{V}'}/TM_z)|_z$ in (6.104.2). This involution is orientation reversing. Also $\tilde{\lambda}$ is antisymmetric under the involution, as follows from its explicit description (or the fact that $\widetilde{L}$ is antisymmetric under the involution of $M \times M$ exchanging the two copies of M). Hence the integrand in (6.104.2) is multiplied by $(-1)^{I'} = -(-1)^{l'}$. For l' odd the integrand is invariant, whereas the orientation is not, so the integral in (6.104.2) vanishes.

Remark. Another way at arriving at the final result for the sum I_l^{conn} over connected graphs $\mathbf{G}$, without using (6.114), is to observe that all of our calculations above for $d_{\mathrm{Met}} I_l$ apply to $d_{\mathrm{Met}} I_l^{\mathrm{conn}}$ if we omit terms coming from disconnected graphs $\mathbf{G}$. To describe this, we need to use the graphical interpretation of the sum (6.100) defining $\overline{C}_{V'',I''}$. We will not elaborate on this now except to say that $\overline{C}_{V'',I''}$ is given by a sum over labeled graphs $\mathbf{G}''$ with I'' edges and with vertices labeled from the set $\{z\} \cup \underline{V}''$. Since the graphs $\mathbf{G}'$ summed over to yield $\overline{A}_{V',I'}$ are connected anyway, the restriction that $\mathbf{G}$ be connected means that the graphs $\mathbf{G}''$ must also be connected. When $E = 0$, the vertex labeled by z is always disconnected from the rest of $\mathbf{G}''$, which must therefore be empty. This means that V'' equals zero. So the only terms that contribute to the anomaly are when $l'' = 0$ and $l' = l$.

Appendix. Graded tensor product and push-forward integrals

Let A^* and B^* be graded algebras over $\mathbb{R}$ with unit. The graded tensor product $A^* \hat{\otimes} B^*$ is the tensor product of the underlying vector spaces of A^* and B^* equipped with the multiplication law given by

$$(A.115)\qquad (a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{|b_1|\,|a_2|}(a_1 a_2) \otimes (b_1 b_2)$$

for $b_1 \in B^{|b_1|}$ and $a_2 \in A^{|a_2|}$ of pure degree, and defined for all b_1, a_2 by linearity. There is a natural graded algebra isomorphism $A^* \hat{\otimes} B^* \cong B^* \hat{\otimes} A^*$

taking $a \otimes b$ to $(-1)^{|a||b|} b \otimes a$ for a, b of pure degree. We write $a \otimes 1$ as a and $1 \otimes b$ as b, and multiplication with or without a wedge product symbol, e.g., $(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1 b_1 a_2 b_2 = a_1 \wedge b_1 \wedge a_2 \wedge b_2$. If $A^* = \Lambda^*(V)$ and $B^* = \Lambda^*(W)$, then there is a natural graded algebra isomorphism $A^* \hat{\otimes} B^* \cong \Lambda^*(V \oplus W)$.

Suppose $\mathbf{B}^* \to Y$ is a bundle of graded algebras over an oriented manifold Y. We define multiplication of forms in $\Omega^*(Y, \mathbf{B}^*)$ by identifying this space with the graded algebra $\Gamma(Y, \Lambda^*(T^*Y) \hat{\otimes} \mathbf{B}^*)$.

When $\mathbf{B}^* = Y \times B^*$ is a trivial bundle, the algebras $\Omega^*(Y) \hat{\otimes} B^*$, $\Omega^*(Y, B^*)$, and $\Omega^*(Y, \mathbf{B}^*)$ are by definition all equal. Our notion of integration over Y of such forms is defined by

$$\text{(A.116)} \qquad \int_Y \omega \wedge b = \left(\int_Y \omega \right) b \in B^* \quad \text{for } \omega \in \Omega^*(Y) \text{ and } b \in B^*$$

and linearity.

Now suppose X, Y are manifolds and $\mathbf{C}^* \to X$ is a bundle of graded algebras. Let $\pi_X : X \times Y \to X$ be the projection map. We may identify $\Omega^*(X \times Y, \pi_Y^*(\mathbf{C}^*))$ with a graded and completed tensor product $\Omega^*(X, \mathbf{C}^*) \hat{\otimes} \Omega^*(Y)$. Identifying $\Omega^*(X, \mathbf{C}^*)$ with the algebra B^* in the last paragraph gives a notion of integration over Y of forms in

$$\text{(A.117)} \qquad \Omega^*(X \times Y, \pi_Y^*(\mathbf{C}^*)) \cong B^* \hat{\otimes} \Omega^*(Y) \cong \Omega^*(Y) \hat{\otimes} B^*.$$

Given $\rho \in \Omega^*(X \times Y, \pi_Y^*(\mathbf{C}^*))$, let $\psi = \int_Y \rho \in \Omega^*(X, \mathbf{C}^*)$. Then we write

$$\text{(A.118)} \qquad \psi(x) = \int_{y \in Y} \rho(x, y) \in \Lambda^*(T^*X_x) \hat{\otimes} \mathbf{C}_x^*, \quad \text{for } x \in X.$$

If $\rho(x, y) = \omega(x) \wedge \eta(y)$, then

$$\begin{aligned} \psi(x) &= \int_{y \in Y} (\omega(x) \wedge \eta(y)) \\ \text{(A.119)} \qquad &= (-1)^{|\omega||\eta|} \int_{y \in Y} (\eta(y) \wedge \omega(x)) \\ &= (-1)^{|\omega||\eta|} \left[\int_{y \in Y} \eta(y) \right] \omega(x). \end{aligned}$$

Note that this sign convention implies that

$$\text{(A.120)} \qquad D_X \int_Y \rho = (-1)^{\dim(Y)} \int_Y D_X \rho,$$

where D_X is covariant exterior derivative operator in the X directions associated to a connection on $\mathbf{C}^*$.

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